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CARTESIAN PLANE GEOMETRY

PART·I·

ANALYTICAL CONICS

BY

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CARTESIAN PLANE GEOMETRY

PART I—ANALYTICAL CONICS

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PART I

ANALYTICAL CONICS

BY

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PREFACE

FOR many years I have had a wish to write a text-book on Elementary Cartesian Geometry, and have promised myself the pleasure at some future time. But that time had always seemed to be in the far distant future until two years ago, when the editor of this series of text-books asked me to take over the task of writing this book, undertaken by R. W. H. T. Hudson, and by his tragic death left unfinished. When I acceded to this request, Mr. Greenstreet kindly sent me the outline that had been drawn up by Mr. Hudson, intimating however that I was not to feel in any way bound by it.

It rarely happens that the plan of one writer can be adopted by another. I have made my own, but have gladly taken suggestions from the MS. thus placed at my disposal, in particular as to the content of the preliminary chapter, as to the proper point at which to introduce change of axes, and as to the slight importance attached to many propositions that are usually emphasised. Also I have drawn encouragement from the opinion there expressed that it is not only admissible, but advisable, to include (even in a school-book) many matters of philosophical interest that are usually excluded, and have ventured to introduce the ideas of the circular points and isotropic (or null) lines.

The distinctive feature of this book is the systematic use of Cartesian line-coordinates, concurrently with point-coordinates, from the very first. This is not from any desire for purely formal symmetry, attractive as this undoubtedly is, but because of my conviction that by this course the subject as a whole is made easier, while its power is immensely increased. Not to be too radical, however, I have shown a certain degree of deference to the established order, and have left the balance of power with point-coordinates.

The suggestion that the condition of incidence of a point (x, y) and a line (ξ, η) should precede any use of the equation of a line, to which the arrangement of Chapter II. is due, was I believe made by Clebsch.

I hope that the examples given will be found appropriate as exercises in the use of the methods, and adequate in the range of theorems displayed. In the selection of stock examples, and the construction of new ones, I have striven to avoid those that suggest the methods of pure geometry rather than of analytical geometry, and also those that are simply exercises in algebraic gymnastics. I do not think that any of those given should be beyond an intelligent student who has reached that point of the book by legitimate means.

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November 1906.

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CHAPTER I

INTRODUCTION

1. IN analytical geometry we make use of the notation and processes of algebra to prove geometrical properties of figures. The subject is geometry; the language is algebraic. Plane analytical geometry, with which we are now concerned, contains the algebraic treatment of plane figures; in solid analytical geometry similar processes are applied to solid figures. It will be found that the application of algebra to geometry is not only a convenient and sure method of proving properties of figures; it is more than this; it offers valuable suggestions as to the nature of these figures. In return, geometry supplies an intelligible interpretation of certain algebraic results, so that both subjects derive advantage from their combination.

2. The figures we are concerned with in plane geometry may be built up from points and straight lines. Either we have a definite number of these elements, whose positions are assigned individually, or we have an indefinite number, whose positions are connected according to some law. Some of the points may lie on a line; they then determine pieces of the line, or *segments*. Some of the

lines may pass through a point; they then determine *angles* about the point. We shall find it necessary to deal with segments and angles, and also with areas. We shall find, too, that each of these magnitudes is susceptible of sign, that is to say, that each may be regarded from two opposite points of view, which can conveniently be distinguished by a positive or negative sign prefixed to the numerical measure of the magnitude.

3. The position of a point P on a line can be assigned by means of its distance from a fixed point O of the line. Not simply by the numerical measure of this distance, for this gives two points, one on each side of O; but by the numerical measure with the sign + or − prefixed. As explained in trigonometry, lengths measured along the line in one specified direction are indicated by positive numbers, and lengths measured in the opposite direction by negative numbers. Thus $PQ = -QP$.

One advantage of attaching sign to these lengths is that certain laws of combination hold for segments, irrespective of the relative position of the points by which the segments are terminated. We have as a universal truth for segments on a line—

$$OP + PQ = OQ,$$

which is equivalent to "the step from O to P, followed by the step from P to Q, gives the same result as the step from O to Q."

In Fig. 1 (*a*) OP and PQ are positive; *e.g.*—

if $\qquad OP = +5,\ PQ = +3$

then $\qquad OQ = +8.$

In Fig. 1 (*b*) OP is positive, PQ is negative; *e.g.*—

if $\qquad\qquad OP = +5,\ PQ = -3,$

then $\qquad\qquad OQ = +2.$

It is indifferent which direction is chosen to be positive. If we choose as indicated in Fig. 1, then all numbers from 0 to $+\infty$ are represented on the line from O towards the right; numbers from 0 to $-\infty$ are represented on the line along the part to the left of O. The positive direction is frequently indicated by an arrowhead, as

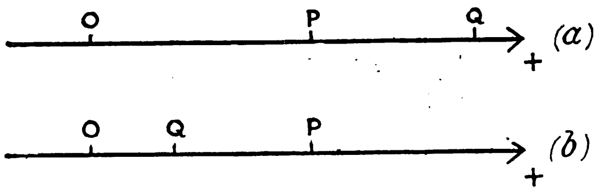


Fig. 1.

shown in the figure. When the positive direction has been assigned, the statement that a point P on the line lies at a given distance from a given point A on the line becomes absolutely definite. A line on which the positive direction has been assigned is called a *directed line.*

4. When a line rotates about a point there are two possibilities: it may rotate either in the same direction as the hands of a clock, or in the opposite direction. This last, counterclockwise as it is called, is usually taken for the positive direction of rotation. The notation employed for angles may be made to correspond to that

used for segments, if lines are denoted by single letters a, b, c, etc., precisely as points are denoted by A, B, C, etc. By the segment AB we mean the distance through which a point, moving along the line that joins the points A, B, must travel to pass from A to B, this distance being taken with the proper sign; by the angle ab we mean the angle through which a line, rotating about the point of intersection of the lines a, b, must rotate to pass from the position a to the position b, this angle being taken with the proper sign. This is often called the angle that b makes with a; in this mode of speaking, we look upon b as the revolving line, and consider that it starts from the position a.[1]

For angles, as for segments—

$$ab = -ba.$$

Also, $$ap + pb = ab,$$

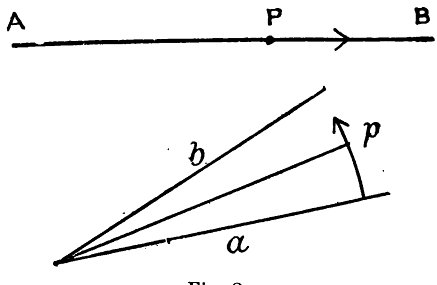


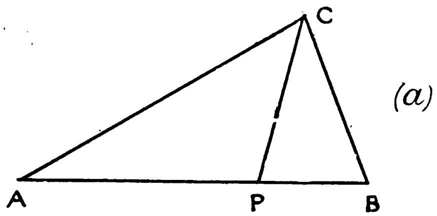
Fig. 2.

which is equivalent to "rotation from a to p, followed by

[1] If a, b are directed lines, the rotation must bring the positive direction on one line to coincidence with the positive direction on the other.

rotation from p to b, gives the same final position as rotation from a to b."

5. At first sight an *area* perhaps does not seem to require any sign. Yet certainly we can add and subtract areas. If three points ABP on a line are joined to a



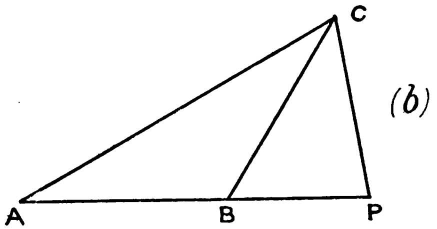


Fig. 3.

point C not on the line, as in Fig. 3, then in (a) the triangles APC, PBC added together make up the triangle ABC, while in (b) subtraction is necessary.

Let h denote the altitude of the triangles. As regards the segments of the line—

$$\mathrm{AP} + \mathrm{PB} = \mathrm{AB} \text{ for both } (a) \text{ and } (b),$$

hence $$\tfrac{1}{2}h\,.\,\mathrm{AP} + \tfrac{1}{2}h\,.\,\mathrm{PB} = \tfrac{1}{2}h\,.\,\mathrm{AB},$$

that is, $$\triangle\,\mathrm{APC} + \triangle\,\mathrm{PBC} = \triangle\,\mathrm{ABC}.$$

This relation in the areas applies to the first figure (a); it holds for (b) also, if we look upon the area PBC as negative in (b), positive in (a).

The difference between (a) and (b) as regards the area PBC is simply this. In (a), when we reach B we turn to the left, that is, in the direction already chosen as the positive direction of rotation; in (b) we turn to the right (Fig. 4). Thus if we attend to the specified direction on

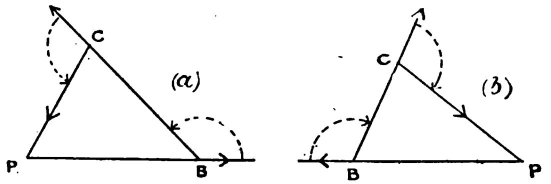


Fig. 4.

the lines (*viz.* PB, BC), and observe the general convention as to the positive direction of rotation, we have the area on the left hand in the first case, on the right hand in the second. We therefore make the following convention as to the sign of an area.

Convention.—The sign of an area is to be taken positive if the area lies on the left hand as we pass round the boundary in the order indicated.

6. If a point P is joined to the vertices of a triangle ABC three triangles are formed whose areas, properly combined by addition and subtraction, make up the area of the given triangle. The point P may lie (1) inside the triangle, (2) in a region separated from the interior of the

triangle by one side, or (3) in a region separated from the interior by two sides (Fig. 5).

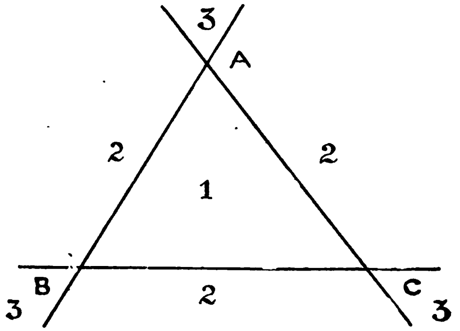


Fig. 5.

Let the numerical measures of the areas ABC, BCP, CAP, ABP (without sign) be $\Delta_0, \Delta_1, \Delta_2, \Delta_3$.

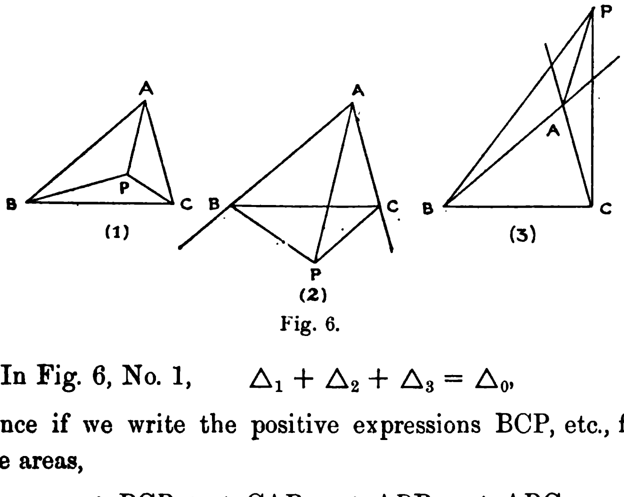


Fig. 6.

In Fig. 6, No. 1, $\Delta_1 + \Delta_2 + \Delta_3 = \Delta_0$,

hence if we write the positive expressions BCP, etc., for the areas,

$$\Delta\,BCP + \Delta\,CAP + \Delta\,ABP = \Delta\,ABC.$$

In 2, $\triangle$ BCP is negative, hence $\triangle \text{BCP} = -\Delta_1$.

Now from the figure, $\Delta_2 + \Delta_3 = \Delta_0 + \Delta_1$,

hence $$\triangle \text{CAP} + \triangle \text{ABP} = \triangle \text{ABC} - \triangle \text{BCP},$$

$$\therefore \triangle \text{BCP} + \triangle \text{CAP} + \triangle \text{ABP} = \triangle \text{ABC}.$$

In 3, $\triangle$ CAP and $\triangle$ ABP are negative, while $\triangle$ BCP is positive; and from the figure,

$$\Delta_1 = \Delta_0 + \Delta_2 + \Delta_3,$$

hence $$\triangle \text{BCP} = \triangle \text{ABC} - \triangle \text{CAP} - \triangle \text{ABP},$$

$$\therefore \triangle \text{BCP} + \triangle \text{CAP} + \triangle \text{ABP} = \triangle \text{ABC}.$$

Thus in every case, thanks to the convention of sign, we have—

$$\triangle \text{BCP} + \triangle \text{CAP} + \triangle \text{ABP} = \triangle \text{ABC}.$$

7. Let A, B be two known points on a line, and P any third point on that line; then if AP is given, BP (or PB) is known. For from the relation AP + PB = AB, we derive PB = AB − AP. It is often convenient to give the ratio of the two segments AP, PB instead of the actual length of either. This is called the ratio in which AB is divided by (or at) P. (Notice carefully the order of the letters; the ratio in which AB is divided is that of AP to PB, while the segment BA is divided by P in the ratio BP : PA.)

The meaning of the phrase is obvious when P lies between the points A, B; the segment AB is then actually divided into two parts at P, and the two parts AP, PB are measured in the same direction. Even when P lies on the line outside the segment we speak of AB as divided by P;

it is divided externally instead of internally. In Fig. 7 (1) the segments AP, PB are measured in the same direction, hence both are positive or both negative; their

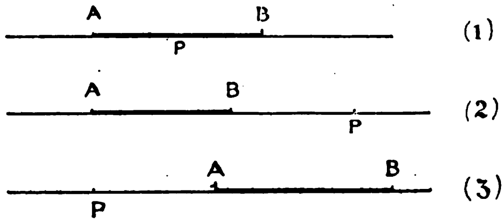


Fig. 7.

ratio is therefore *positive*. In (2), (3), AP, PB are measured in opposite directions, and their ratio is consequently *negative*. In (2), AP : PB is a negative quantity whose numerical value is greater than unity; in (3) this ratio is a negative quantity numerically less than unity.

Denote the ratio by r; every value of r, positive or negative, is associated with one point of the line; an indefinitely small change in the value of r produces an indefinitely small alteration in the position of P. As P

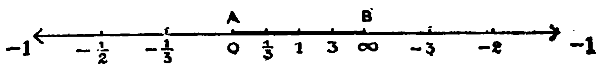


Fig. 8.

moves from A to B, r increases from 0, passes through all positive values, and attains the value $+\infty$ when P reaches B (for AP : PB is then AB : 0, *i.e.* ∞). As P moves from A (Fig. 8) towards the left, r passes through negative

values lying between 0 and -1; when r attains the value -1, P is at an indefinitely great distance. For—

$$\begin{aligned} \mathrm{AP}:\mathrm{PB} = \frac{\mathrm{AP}}{\mathrm{PB}} &= -\frac{\mathrm{PA}}{\mathrm{PB}} \\ &= -\frac{\mathrm{PB}-\mathrm{AB}}{\mathrm{PB}} \\ &= -1+\frac{\mathrm{AB}}{\mathrm{PB}}, \end{aligned}$$

hence if PB is infinitely great, that is, if $\frac{\mathrm{AB}}{\mathrm{PB}} = 0$, the ratio $\mathrm{AP}:\mathrm{PB} = -1$. The point P is then said to lie "at infinity." Similarly if P moves from B towards the right, r changes from an infinitely great negative quantity through smaller negative values, tending towards -1 as P moves further away. For—

$$\begin{aligned} \frac{\mathrm{AP}}{\mathrm{PB}} = -\frac{\mathrm{AP}}{\mathrm{BP}} &= -\frac{\mathrm{AB}+\mathrm{BP}}{\mathrm{BP}} \\ &= -\frac{\mathrm{AB}}{\mathrm{BP}} - 1. \end{aligned}$$

Hence $\frac{\mathrm{AP}}{\mathrm{PB}} = -1$ if $\frac{\mathrm{AB}}{\mathrm{BP}} = 0$, that is, if BP is indefinitely great. The point P lies "at infinity" towards the right.

8. There are two important matters to notice here. First, the point at infinity on the right is the same point as the point at infinity on the left. For the quantity r, on whose value the position of P depends, passes steadily over all negative values from 0 to $-\infty$; a value of r just before the value -1 is indefinitely little removed from a value of r just after the value -1, and

thus the position of P as r approaches -1 is very slightly different from the position of P as r leaves -1. To explain this conclusion, that P arrives at the same point whether it travels along the line in one direction or the other, we have to look upon the line as a closed figure, continuous through infinity; so that if we start from any point A and travel in one direction we arrive again at A having passed through the point at infinity.

Secondly, the point P arrives at B, from A, with a value of r that is positive and very great; it leaves B, on the right, with a value of r that is negative and very great. These numerical magnitudes, a very great positive quantity and a very great negative quantity, must be regarded as close together in value; $+\infty$ is the same as $-\infty$. This idea has already presented itself in trigonometry; if θ is a right angle, then $\tan\theta = +\infty$ or $-\infty$ according as θ approaches the right angle from one side or the other. Algebraically, inasmuch as $\frac{1}{x}$ increases indefinitely when x is indefinitely diminished, which we write symbolically in the form $\frac{1}{0} = \infty$, we see that $+\infty$ is the same as $-\infty$ in precisely the sense in which $+0$ is the same as -0, where 0 is the symbol for an indefinitely small quantity. Thus both algebraically and geometrically, opposite infinities are the same.

9. When the position of P is determined by the value of r (= AP : PB) the point at infinity is given by $r = -1$, the point of bisection of AB by $r = +1$; the segment AB

is divided in the same ratio, but internally in the second case, externally in the first. This is a particular case of a general relation. We may divide the segment AB both internally and externally in any other ratio, thus obtaining two points P, Q, which are said to divide AB harmonically; the points P, Q are also said to be harmonic conjugates with respect to the points A, B.

From the definition of the harmonic relation—

$$\frac{AP}{PB} = -\frac{AQ}{QB},$$

$$\therefore \frac{AP}{PB} \cdot \frac{QB}{AQ} = -1,$$

$$i.e. \ \frac{AP}{AQ} \cdot \frac{QB}{PB} = -1,$$

hence
$$\frac{-PA}{AQ} \cdot \frac{-BQ}{PB} = -1,$$

$$i.e. \ \frac{PA}{AQ} \cdot \frac{BQ}{PB} = -1,$$

$$\therefore \frac{PA}{AQ} = -\frac{PB}{BQ},$$

that is, A, B divide PQ harmonically. Thus we have proved that if P, Q are harmonic conjugates with respect to A, B, then A, B are harmonic conjugates with respect to P, Q; the harmonic relation therefore involves the two pairs of points AB, PQ symmetrically.

Examples.

1. Insert, on the line determined by A, B, the points for which $r = \pm 2, \pm 4, \pm \frac{1}{2}, \pm \frac{4}{5}$.

2. What is the harmonic conjugate to the point of bisection of the segment whose extremities are given by $r = 4$, $r = 5$?

3. If PQ divide AB harmonically, prove that AB is the "harmonic mean" between AP and AQ, by showing that $\frac{2}{AB} = \frac{1}{AP} + \frac{1}{AQ}$.

4. The segment AB is trisected at P_1, P_2; the harmonic conjugates to P_1, P_2 with respect to AB are Q_1, Q_2. If AB is 9 inches, how far apart are Q_1, Q_2?

5. Lines are drawn to bisect the vertical angle of a triangle internally and externally. Prove that the base of the triangle is divided harmonically.

6. AB is divided harmonically at P, Q, and bisected at O. Prove $OA^2 = OP \cdot OQ$.

10. In elementary geometry parallel lines are defined as lines that never meet, however far produced. This

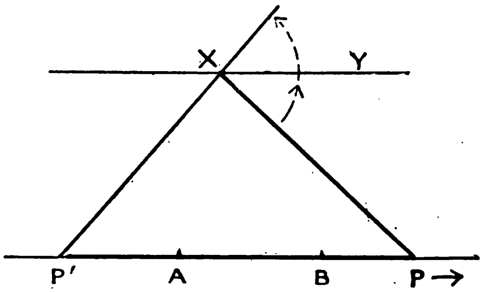


Fig. 9.

does not assert that the lines do not meet at infinity, for however far a line may be produced, the point at infinity is still indefinitely remote. We shall now give reasons for the statement that parallel lines meet at infinity. Let X be any point on the line XY parallel to AB. Take a point P on the line AB at a finite distance to the right of the segment AB (Fig. 9), join XP, then XP is not parallel to AB. Similarly if P′ is at any finite distance to the left of AB, XP′ is not

parallel to AB. Now let P start from A, and trace the whole line, arriving finally at A again. The line XP is initially inclined towards AB; it rotates in the direction marked, and after passing over all intermediate positions arrives at such a position as XP′; XY is one of these intermediate positions. Now since XY is parallel to AB, it does not meet AB at a finite point to the right, nor at a finite point to the left. Hence when XP coincides with XY, P must be the point at infinity; for the entire line AB consists of the finite segment AB, finite points to the right of this segment, finite points to the left of the segment, and the point at infinity. Thus the line XY, parallel to AB, may be regarded as meeting AB at infinity.

CHAPTER II

COORDINATES OF POINT AND LINE

11. THE figures we are concerned with in plane geometry are built up of points and straight lines, placed in proper position; in order to apply algebraic processes to these figures we represent the positions of points and lines by means of numbers. The position of a point is determined by its distances from two intersecting directed lines, the axes, which are most conveniently taken to be at right angles. The position of P with regard to the axes Ox, Oy, is known when MP and NP (Fig. 10) are known. Since

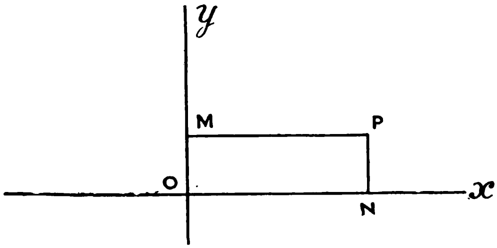


Fig. 10.

MP = ON, it suffices to use the simpler diagram, Fig. 11; the position of P is then assigned by means of ON and NP, the *coordinates* of P. The point O is the origin of coordinates; Ox, Oy are the axes of coordinates, Ox is the

axis of x, Oy the axis of y; NP is the *ordinate* of P; ON, the part of the axis of x starting from O that is cut off by

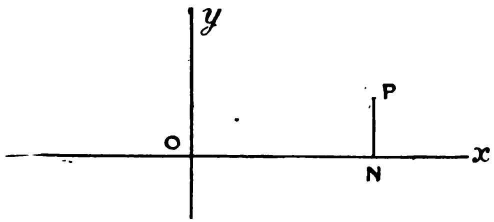


Fig. 11.

the ordinate, is the *abscissa* of P. These coordinates are denoted by x (ON or its equal MP) and y (NP or its equal OM). Thus x is measured along or parallel to the axis of x, y along or parallel to the axis of y. The distance of P from the axis of x is y, its distance from the axis of y is x.

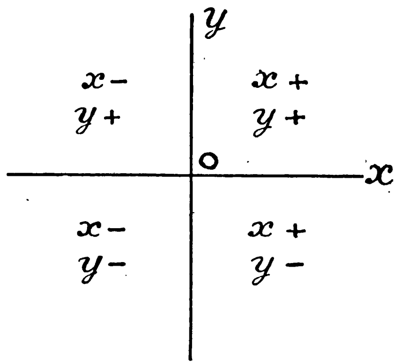


Fig. 12.

The axis of x is usually taken to be horizontal, with the positive direction to the right; on the axis of y the positive direction is upwards, so that rotation through a right angle in the positive direction takes us from the positive direction on the axis of x to the positive direction on the axis of y. The four quadrants into which the axes divide the whole plane are distinguished by the signs of the coordinates,

as shown in Fig. 12. A point whose coordinates are $x = 4$, $y = 3$ is spoken of as the point (4, 3), it being understood that the coordinate first mentioned[1] is x. In Fig. 13 are represented, on paper ruled in unit squares, the points A (3, 2), B (2, 5), C (− 2, 4), D (−3, 0), E (− 2, −1), F (0, −2), G (2, −1), H (4, 0).

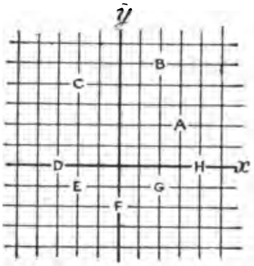


Fig. 13.

The coordinates of the origin are (0, 0). On the axis of x all points have $y = 0$, thus any point on OX is $(a, 0)$. Similarly any point on OY is $(0, b)$.

12. The distance from the origin to the point (x, y) is

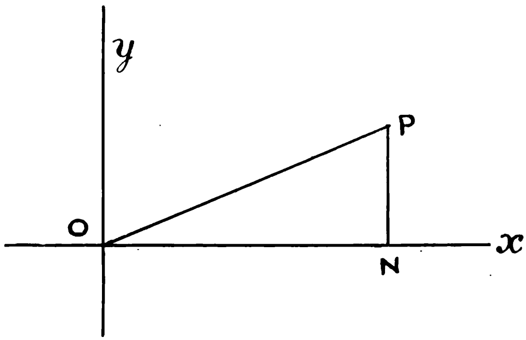


Fig. 14.

$\sqrt{x^2 + y^2}$. For ONP (Fig. 14) is a right-angled triangle,

[1] In problems in which the numerical values of quantities, supposed known, are represented by letters, the coordinates of given points are represented by pairs of letters such as (a, b), (h, k), etc.; or by x, y with suffixes (x_1, y_1) (x_2, y_2), or with accents (x', y') (x'', y'').

hence
$$\begin{aligned} OP^2 &= ON^2 + NP^2 \\ &= x^2 + y^2, \\ \therefore OP &= \sqrt{x^2 + y^2}. \end{aligned}$$

Note.—When the sign is to be attended to, the distance from the origin to any point P is usually taken to be positive.

The distance from any point P (x_1, y_1) to any other point Q (x_2, y_2) can be expressed in terms of the coordinates of the two points. Draw the ordinates NP, RQ (Fig. 15);

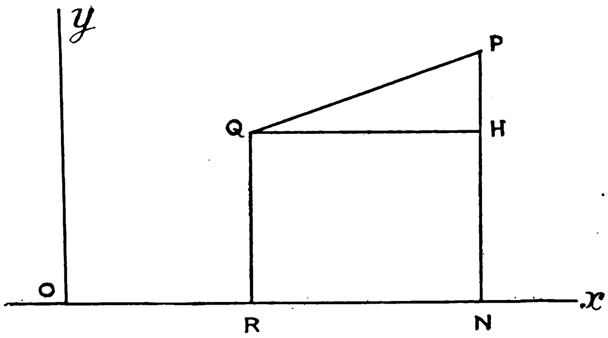


Fig. 15.

draw QH parallel to Ox, to meet NP in H. Then QHP is a right-angled triangle, hence

$$PQ^2 = QH^2 + HP^2.$$

Now $\quad QH = RN = ON - OR = x_1 - x_2,$

and $\quad HP = NP - NH = NP - RQ = y_1 - y_2,$

hence $\quad PQ^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2,$

$$\therefore PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

For example, the distance between (3, 1) and (– 2, 4) is—

$$\sqrt{(3+2)^2+(1-4)^2} = \sqrt{25+9} = \sqrt{34}.$$

Note.—If we simply wish to give the distance apart of the points P, Q, the numerical value of this expression is all that is needed; no sign is then attached. If however sign is to be attended to, PQ is of one sign, QP of the opposite sign.

EXAMPLES.

1. Mark on a diagram the points (2, 1), (6, 0), (–3, 2), (3, 4) (0, 3), (–4, –5).
2. Find the distance from (2, 1) to each of these points.
3. Find the distance from O to each of these points.

13. A straight line that is not parallel to either axis, and does not pass through the origin, meets the axes at points A, B; the line is said to make intercepts a, b on the axes, where OA $= a$, OB $= b$. The directed values of a, b do determine the position of the line, but it is

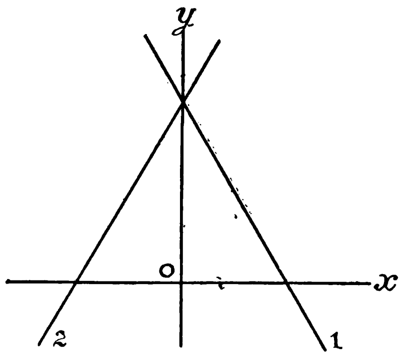


Fig. 16.

found preferable to use as coordinates the negative reciprocals of a, b, namely, $-\frac{1}{a}$ and $-\frac{1}{b}$; these we shall

denote by ξ, η. The two lines 1, 2 in Fig. 16 have coordinates $(-1, -\frac{1}{2})$, $(1, -\frac{1}{2})$.

If a line is parallel to the axis of x, a is infinite, hence $\xi = 0$; if parallel to the axis of y, b is infinite, hence $\eta = 0$. A line through O does not lend itself conveniently to this mode of representation; a and b are both zero, hence $\xi = \infty$ and $\eta = \infty$. If $\xi = 0$ and $\eta = 0$, both intercepts on the axes are infinite; the line lies entirely at infinity. If $\xi = 0$ while η is finite, the line is parallel to the axis of x; and if $\eta = 0$ while ξ is finite, the line is parallel to the axis of y.

14. The distance p from O to a line (ξ, η) is $\dfrac{1}{\sqrt{\xi^2 + \eta^2}}$.

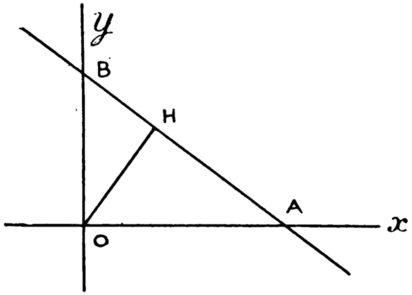


Fig. 17.

For in Fig. 17,

$$\triangle \text{OAB} = \tfrac{1}{2}\text{OA}\,.\,\text{OB} = \tfrac{1}{2}ab\,;$$

also $$\triangle \text{OAB} = \tfrac{1}{2}\text{OH}\,.\,\text{AB} = \tfrac{1}{2}\text{OH}\sqrt{a^2 + b^2}.$$

Hence $$ab = \text{OH}\sqrt{a^2 + b^2},$$

$$\therefore\ \text{OH} = \frac{ab}{\sqrt{a^2 + b^2}} = \frac{\dfrac{1}{\xi\eta}}{\sqrt{\dfrac{1}{\xi^2} + \dfrac{1}{\eta^2}}} = \frac{1}{\sqrt{\xi^2 + \eta^2}}.$$

When it is found convenient to treat as a directed line the line through O at right angles to any line AB, the positive direction is taken to be *from* the origin *towards* the line. In accordance with this convention the distance from O to any line is to be taken positive; hence

$$p = \frac{1}{\sqrt{\xi^2 + \eta^2}}.$$

In order that positive directions on parallel lines may be the same, the distance from any other point to the given line is taken to be *positive* if the point and the origin lie on the same side of the line, *negative* if the point and the origin are separated by the line. In Fig. 18 the distances from P_1 P_2 to the lines shown are positive; the distances from the points Q to the lines are negative.

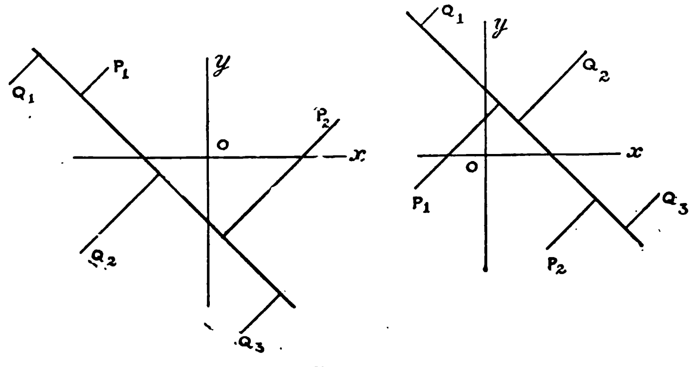


Fig. 18.

Note.—The distance from the line to the point is numerically the same as the distance from the point to the line, but opposite in sign.

15. Denote the distance from (x, y) to (ξ, η) by d, so that d stands for a length with a sign prefixed. We proceed to prove that—

$$\frac{d}{p} = \xi x + \eta y + 1, \text{ or } d = \xi x + \eta y + 1 \div \sqrt{\xi^2 + \eta^2}.$$

Let AB be the line (ξ, η), P the point (x, y) (Fig. 19).

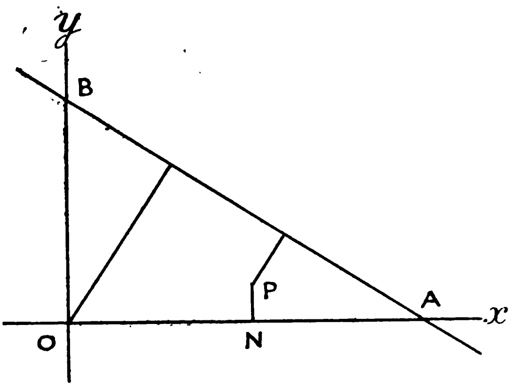


Fig. 19.

By joining P to the points O, A, B we form triangles, whose areas satisfy a relation (§ 6) for all positions of P,

$$\triangle \text{OAP} + \triangle \text{ABP} + \triangle \text{BOP} = \triangle \text{OAB},$$

$$\therefore \frac{\triangle \text{OAP}}{\triangle \text{OAB}} + \frac{\triangle \text{ABP}}{\triangle \text{OAB}} + \frac{\triangle \text{BOP}}{\triangle \text{OAB}} = 1.$$

Now since OAP and OAB are on the same base OA, their numerical ratio is that of their altitudes NP, OB; and since the ratio is positive if P and B lie on the same side of OA, that is, if y and b have the same sign (and otherwise negative), this ratio is completely represented by $\frac{y}{b}$.

Hence $$\frac{\triangle \text{OAP}}{\triangle \text{OAB}} = \frac{y}{b}.$$

Similarly $$\frac{\triangle \text{BOP}}{\triangle \text{OAB}} = \frac{\triangle \text{BOP}}{\triangle \text{BOA}} = \frac{x}{a},$$

and $$\frac{\triangle \text{ABP}}{\triangle \text{OAB}} = \frac{\triangle \text{ABP}}{\triangle \text{ABO}} = \frac{d}{p}.$$

Hence $$\frac{y}{b} + \frac{d}{p} + \frac{x}{a} = 1,$$

i. e. $$-\frac{1}{a}\cdot x - \frac{1}{b}\cdot y + 1 = \frac{d}{p},$$

or $$\xi x + \eta y + 1 = \frac{d}{p} = d\sqrt{\xi^2 + \eta^2}.$$

An alternative proof is the following—

Let $\angle$ AOH $= \alpha$. Draw from N, P, lines perpendicular to OH,

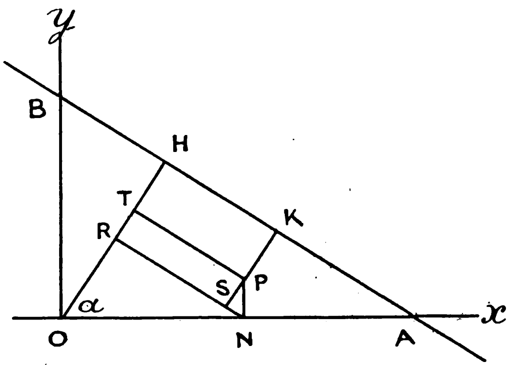


Fig. 20.

viz. NR, PT, and from P, PS perpendicular to NR, meeting AB in K. Then $\angle$ PNS $= \alpha$ (Fig. 20).

Now
$$\mathrm{OH} = \mathrm{OR} + \mathrm{RT} + \mathrm{TH};$$
$$\mathrm{OH} = p,\ \mathrm{OR} = \mathrm{ON} \cos a = x \cos a,$$
$$\mathrm{RT} = \mathrm{SP} = \mathrm{NP} \sin a = y \sin a,$$
$$\mathrm{TH} = \mathrm{PK} = d.$$
$$\therefore p = x \cos a + y \sin a + d.$$

Now
$$\cos a = \frac{\mathrm{OH}}{\mathrm{OA}} = \frac{p}{a} = -p\xi,$$
$$\sin a = \frac{\mathrm{OH}}{\mathrm{OB}} = \frac{p}{b} = -p\eta,$$
$$\therefore p = -p\xi x - p\eta y + d.$$
$$\therefore p\xi x + p\eta y + p = d,$$
$$\therefore \xi x + \eta y + 1 = \frac{d}{p}.$$

The result applies to all cases, as will be seen on constructing the different diagrams and examining the applicability of the proof.

Examples.

1. Find with their proper signs the distances from the points $(5, 2), (5, -2), (-5, 2)$ to the lines $(-\frac{1}{8}, -\frac{1}{12}), (\frac{1}{8}, 1)$.

2. Represent these points and lines on a diagram, and use this to verify your statements as to signs.

16. Since p, or $\dfrac{1}{\sqrt{\xi^2 + \eta^2}}$, is positive, and d is positive or negative according to the side of the line (ξ, η) on which the point (x, y) lies, we see that if $\xi x + \eta y + 1$ is positive, the point (x, y) lies on the same side of the line (ξ, η) as the origin; while if $\xi x + \eta y + 1$ is negative, the point (x, y) lies on the side of (ξ, η) that is remote from the origin. Thus e.g. if $x, y = 7, 8$, and $\xi, \eta = -5, 3$, we know without referring to a diagram that the point lies on the side of the line remote from the origin; for $\xi x + \eta y + 1 = -35 + 24 + 1$, which is negative.

EXAMPLE.

Find on which side of each of the lines $(-1, -\frac{1}{2})$, $(1, \frac{1}{2})$ the points $(\frac{1}{2}, \frac{1}{2})$, $(1, \frac{1}{2})$, $(\frac{1}{2}, 2)$, $(-2, 1)$ lie.

17. The formula proved above, namely—

$$\xi x + \eta y + 1 = \frac{d}{p} = d\sqrt{\xi^2 + \eta^2},$$

is very important. As we have seen, it gives the distance from the point (x, y) to the line (ξ, η) both in magnitude and sign. Also, it shows that the point lies on the line (the line passes through the point) if $\xi x + \eta y + 1 = 0$. For the point is on the line if its distance from the line vanishes, that is, if $\xi x + \eta y + 1 = 0$; and conversely, if $\xi x + \eta y + 1 = 0$, then $\frac{d}{p} = 0$, which gives necessarily $d = 0$, as p is not infinite unless the line lies entirely at infinity. This may be expressed in the form:—the *condition of incidence* of the point (x, y) and the line (ξ, η) is $\xi x + \eta y + 1 = 0$. For example, the condition that (x, y) lie on the line $(3, -4)$ is $3x - 4y + 1 = 0$; the condition that (ξ, η) pass through the point $(2, 5)$ is $2\xi + 5\eta + 1 = 0$.

18. If a line passes through two given points (x_1, y_1), (x_2, y_2), its coordinates satisfy the two equations—

$$\xi x_1 + \eta y_1 + 1 = 0, \ \xi x_2 + \eta y_2 + 1 = 0;$$

hence to find the line, we solve these simultaneous simple equations for ξ, η, obtaining—

$$\xi = \frac{y_1 - y_2}{x_1 y_2 - x_2 y_1}, \ \eta = -\frac{x_1 - x_2}{x_1 y_2 - x_2 y_1}.$$

Example.—Find the line through (5, 3), (– 2, 4).
The coordinates must satisfy—

$$5\xi + 3\eta + 1 = 0,$$
$$-2\xi + 4\eta + 1 = 0.$$
$$\therefore 20\xi + 12\eta + 4 = 0,$$
$$-6\xi + 12\eta + 3 = 0.$$
$$\therefore 26\xi + 1 = 0. \quad \text{Hence } \xi = -\frac{1}{26},$$
$$\eta = -\frac{7}{26};$$

the line has coordinates $-\frac{1}{26}, -\frac{7}{26}$.

Similarly the point of intersection of two lines (ξ_1, η_1), (ξ_2, η_2) is found by solving for x, y the simultaneous simple equations—

$$\xi_1 x + \eta_1 y + 1 = 0, \ \xi_2 x + \eta_2 y + 1 = 0;$$

the coordinates of the point are therefore—

$$x = \frac{\eta_1 - \eta_2}{\xi_1\eta_2 - \xi_2\eta_1}, \ y = -\frac{\xi_1 - \xi_2}{\xi_1\eta_2 - \xi_2\eta_1}.$$

Examples.

1. Find the coordinates of the line through (4, 1), (– 5, – 2). Also of the line through (8, 0), (6, 4).
2. Find the point of intersection of these two lines.
3. Find the coordinates of the lines that join the following pairs of points: (i) (7, 2), (1, 5); (ii) (3, 6), (3, 3); (iii) (– 1, 4), (5, 4).
4. Show that these three lines meet in one point.

19. If the coordinates of two lines satisfy the relation $\xi_1\eta_2 - \xi_2\eta_1 = 0$, that is, if $\frac{\xi_1}{\eta_1} = \frac{\xi_2}{\eta_2}$, the values found above for x, y become $\frac{\eta_1 - \eta_2}{0}$, $\frac{\xi_1 - \xi_2}{0}$, which are infinite unless $\xi_1 = \xi_2$, $\eta_1 = \eta_2$. These equalities, however, cannot

hold, for they would make the two lines identical. Thus the two lines meet at infinity, that is, are parallel, if

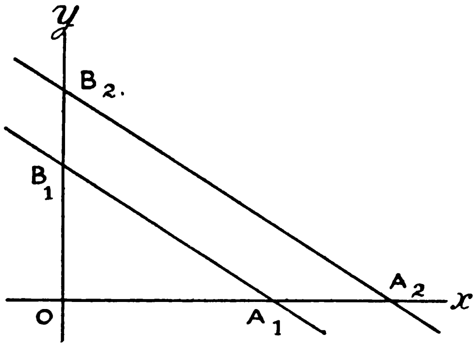


Fig. 21.

$\frac{\xi_1}{\eta_1} = \frac{\xi_2}{\eta_2}$, which gives $\frac{b_1}{a_1} = \frac{b_2}{a_2}$, and thus agrees with the conclusions of elementary geometry (Fig. 21).

Again, if $x_1y_2 - x_2y_1 = 0$, that is, if $\frac{x_1}{y_1} = \frac{x_2}{y_2}$, the values

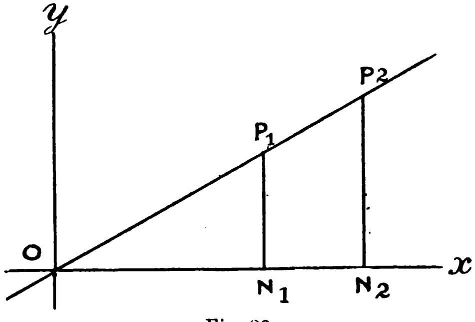


Fig. 22.

found for ξ, η become infinite, indicating that the line ξ, η passes through O; and it is clear from a diagram

(Fig. 22) that the condition that P_1P_2 pass through O, which makes the triangles ON_1P_1, ON_2P_2 similar, is—

$$ON_1 : N_1P_1 = ON_2 : N_2P_2,$$

$$\text{i.e. } \frac{x_1}{y_1} = \frac{x_2}{y_2}.$$

20. If three points lie on one line, they are said to be collinear; if three lines meet in one point, they are said to be concurrent. The condition that the three points (x_1, y_1), (x_2, y_2), (x_3, y_3) be collinear may be obtained by expressing that the point (x_3, y_3) lies on the line determined by (x_1, y_1), (x_2, y_2), whose coordinates are $\dfrac{y_1 - y_2}{x_1y_2 - x_2y_1}$, $-\dfrac{x_1 - x_2}{x_1y_2 - x_2y_1}$. The condition is therefore—

$$x_3\frac{y_1 - y_2}{x_1y_2 - x_2y_1} - y_3\frac{x_1 - x_2}{x_1y_2 - x_2y_1} + 1 = 0,$$

$$\text{i.e. } x_3(y_1 - y_2) + y_3(x_2 - x_1) + x_1y_2 - x_2y_1 = 0,$$

or

$$x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3 + x_1y_2 - x_2y_1 = 0,$$

which may be written also in the form—

$$x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0,$$

or, with sign changed—

$$y_1(x_2 - x_3) + y_2(x_3 - x_1) + y_3(x_1 - x_2) = 0.$$

These conditions are more briefly written—

$$\Sigma(x_2y_3 - x_3y_2) = 0,\ \Sigma x_1(y_2 - y_3) = 0,\ \Sigma y_1(x_2 - x_3) = 0.$$

Similarly we find that three lines (ξ_1, η_1) (ξ_2, η_2), (ξ_3, η_3) are concurrent if $\Sigma(\xi_2\eta_3 - \xi_3\eta_2) = 0$, which can also be written in either of the forms—$\Sigma\xi_1(\eta_2 - \eta_3) = 0$, $\Sigma\eta_1(\xi_2 - \xi_3) = 0$.

Note.—These conditions can be obtained more neatly by the use of determinants. If the points are collinear, lying on (ξ, η), the three equations

$$\xi x_1 + \eta y_1 + 1 = 0$$
$$\xi x_2 + \eta y_2 + 1 = 0$$
$$\xi x_3 + \eta y_3 + 1 = 0$$

can be satisfied. Hence the result of eliminating (ξ, η) must vanish,

$$\text{i.e. } \begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix} = 0;$$

and similarly the condition that three lines be concurrent is—

$$\begin{vmatrix} \xi_1, & \eta_1, & 1 \\ \xi_2, & \eta_2, & 1 \\ \xi_3, & \eta_3, & 1 \end{vmatrix} = 0.$$

21. In Chapter I. (§ 7) it was stated that the position of a point P on a line through two points A, B may be conveniently expressed by means of the ratio AP : PB.

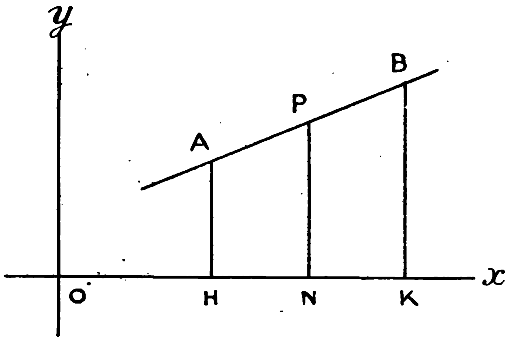


Fig. 23.

Let the coordinates of A, B be (x_1, y_1), (x_2, y_2), and let this ratio be r; the coordinates of P, namely (x, y), can be expressed in terms of x_1, y_1, x_2, y_2, r. For since the ordinates HA, KB, NP (Fig. 23) are parallel,

$$HN : NK = AP : PB.$$

$$\therefore \frac{HN}{NK} = r.$$

Now $$HN = ON - OH = x - x_1,$$

and $$NK = OK - ON = x_2 - x.$$

$$\therefore \frac{x - x_1}{x_2 - x} = r,$$

$$\text{i.e. } x - x_1 = r x_2 - r x,$$

$$\text{or } x(1 + r) = x_1 + r x_2.$$

Hence $$x = \frac{x_1 + r x_2}{1 + r},$$

and similarly $$y = \frac{y_1 + r y_2}{1 + r}.$$

If the ratio AP : PB is given in the form $k : l$, these become— $$x = \frac{l x_1 + k x_2}{l + k}, \quad y = \frac{l y_1 + k y_2}{l + k}.$$

As explained in Chapter I (§ 7), external division of

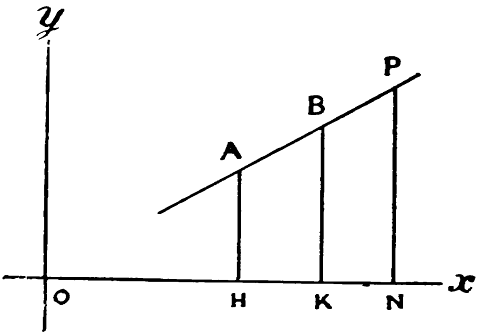


Fig. 24.

the segment AB is indicated by a negative value of r. We have then as before (Fig. 24)—

$$\frac{\mathrm{AP}}{\mathrm{PB}} = r,$$

hence
$$\frac{\mathrm{HN}}{\mathrm{NK}} = r;$$

Now
$$\mathrm{HN} = x - x_1,$$
$$\mathrm{NK} = \mathrm{OK} - \mathrm{ON} = x_2 - x,$$

(notice the direction in which NK is measured); hence, as before,

$$\frac{x - x_1}{x_2 - x} = r; \quad x = \frac{x_1 + rx_2}{1 + r},$$
$$y = \frac{y_1 + ry_2}{1 + r}.$$

Note.—Careful attention should be paid to the form of the result; the segments that have an extremity at (x_1, y_1), (x_2, y_2) are in the ratio $r : 1$ or $k : l$, but in the formula, the r, 1 or k, l go with (x_2, y_2) and (x_1, y_1).

The point of bisection is obtained by writing 1 for r; hence $x = \frac{x_1 + x_2}{2}$, $y = \frac{y_1 + y_2}{2}$. The point at infinity is obtained by writing -1 for r.

The *mean centre* of a set of n points (x_1, y_1), (x_2, y_2), . . . (x_n, y_n) is the point whose coordinates are—

$$\frac{1}{n}(x_1 + x_2 + \ldots + x_n), \frac{1}{n}(y_1 + y_2 + \ldots + y_n),$$

which are usually written $\frac{1}{n}\Sigma x$, $\frac{1}{n}\Sigma y$.

EXAMPLES.

1. Find the points of bisection of the sides of the triangle whose vertices are at (7, 2), (3, 6), ($-$ 1, 4).

2. The line joining each vertex of the triangle in Example 1 to the

point of bisection of the opposite side is divided in the ratio 2 : 1. Show that the point thus obtained is the same for all three lines, and find its coordinates.

3. The vertices of a triangle are (x_1, y_1), (x_2, y_2), (x_3, y_3). Prove that the lines that join the vertices to the points of bisection of the opposite sides meet at the point whose coordinates are—

$$\tfrac{1}{3}(x_1 + x_2 + x_3),\ \tfrac{1}{3}(y_1 + y_2 + y_3).$$

Note.—This point is called the *centroid* of the triangle. It is the mean centre of the three vertices.

4. An arbitrary line through O, the mean centre of a set of points A, B, C, etc., makes with OA, OB, OC, etc., angles α, β, γ, etc. Prove $\Sigma \mathrm{OA} \cos \alpha = 0$.

[Suggestion. If P be any point on the line, use formulæ of § 12 to prove—

$$\Sigma (\mathrm{OP}^2 + \mathrm{OA}^2 - \mathrm{AP}^2) = 0.]$$

5. A point moves so that the sum of the squares of its distances from a number of points A, B, C, etc., has a constant value. Prove that it remains at a constant distance from the mean centre. What is the path of the point?

[Suggestion. Place the origin at the mean centre.]

6. A point P (x, y) lies on the line that joins the origin to the point A (a, b), and $\mathrm{OP} = k\,.\,\mathrm{OA}$. Show that $x = ka$, $y = kb$.

Express algebraically the geometrical statements in Examples 7 to 17. Pay attention to *sign*.

7. $\mathrm{PC} = 5$, where P is (x, y), C is $(3, 4)$.

8. The distance of P from a point on Oy at a distance $+ 4$ from O is equal to the distance of P from the axis of y.

9. The product of the distances of the point (x, y) from the axes $= 9$.

10. The sum of the distances of the point (x, y) from the axes $= 5$.

11. The distance of P (x, y) from the line $(12, 5)$ is equal to $5 \times$ distance of P from the line $(3, 4)$.

12. The distance of the line (ξ, η) from the point $(3, 4) = 5$.

13. The product of the distances of the line (ξ, η) from the points $(4, 0)$, $(-4, 0) = 9$.

14. The product of the distances of the line (ξ, η) from the points $(4, 0)$, $(-4, 0) = -9$.

15. The sum of the distances of (ξ, η) from the points $(5, 2)$, $(-5, -2) = 4$.

16. The difference of the distances of (ξ, η) from the points $(5, 2)$, $(-5, -2) = 1$.

17. The distance from the origin to the point $P = 0$ (*i.e.* $OP = 0$).

18. Show that the area of the trapezium AHKB (Fig. 24) is—

$$\tfrac{1}{2}(x_2y_2 - x_1y_1) + \tfrac{1}{2}(x_2y_1 - x_1y_2).$$

19. Hence show that the area of the triangle OAB is $\frac{1}{2}(x_1y_2 - x_2y_1)$.

20. Show that the area of the triangle whose vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3) is $\frac{1}{2}\Sigma(x_2y_3 - x_3y_2)$. (Notation of § 20.)

21. Find the area of a triangle whose vertices are $(2, 3)$, $(7, -4)$, $(-2, -5)$.

Note.—The area of a triangle is more easily calculated from one of the equivalent expressions—

$$\tfrac{1}{2}\Sigma x_1(y_2 - y_3), \quad -\tfrac{1}{2}\Sigma y_1(x_2 - x_3).$$

All these expressions are written so as to give a positive value for the area if it lies on the left hand as we pass round the boundary in the order indicated. (Convention, § 5.)

22. The coordinates that have here been explained form *one* of many possible sets; they are called Cartesian coordinates, after Descartes (or Des Cartes), who was the first to develop a systematic theory of their use. Any other set of two quantities by means of which the position of a point, or line, in a plane can be determined may be used as coordinates of the point, or line. For some purposes the *polar* coordinates of a point are convenient. These are (1) the radius vector, that is, the distance from the origin O (the pole) to the point P; this is denoted by r, and is usually taken to be positive; (2) the vectorial angle, θ, which is the angle that OP makes with the initial line Ox, a line through O.

If Ox be the axis of x in a Cartesian system, with O

as origin, the relations between the two sets of coordinates

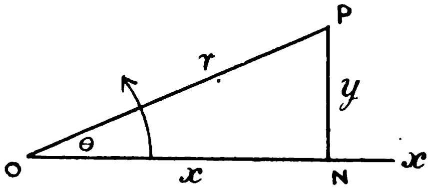


Fig. 25.

can be written down from the right-angled triangle ONP (Fig. 25). We have—

$$\cos\theta = \frac{x}{r},\ \sin\theta = \frac{y}{r};$$

hence $\qquad x = r\cos\theta,\ y = r\sin\theta;\ r = \sqrt{x^2 + y^2}.$

Also $\qquad \theta = \tan^{-1}\dfrac{y}{x};$

$$\cos\theta = \frac{x}{\sqrt{x^2 + y^2}},\ \sin\theta = \frac{y}{\sqrt{x^2 + y^2}}.$$

Similarly the position of a line can be indicated by

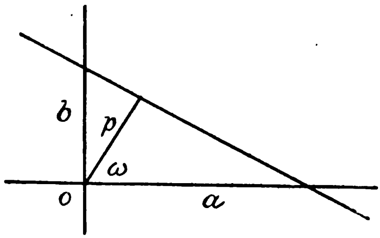


Fig. 26.

means of the length, p, of the perpendicular from the origin, and the angle ω which this makes with Ox. From Fig. 26,

$$a\cos\omega = p, \quad b\sin\omega = p.$$

Hence $\quad -\frac{1}{\xi}\cos\omega = p,\ -\frac{1}{\eta}\sin\omega = p,$

$$i.e.\ \xi = -\frac{\cos\omega}{p},\ \eta = -\frac{\sin\omega}{p},$$

from which, since $p = \frac{1}{\sqrt{\xi^2+\eta^2}}$,

$$\cos\omega = -\frac{\xi}{\sqrt{\xi^2+\eta^2}},\ \sin\omega = -\frac{\eta}{\sqrt{\xi^2+\eta^2}}.$$

Note.—If we write q for $-\frac{1}{p}$, these take a form more in correspondence with the results obtained above for the coordinates of a point, namely,

$$\xi = q\cos\omega,\ \eta = q\sin\omega;\ q = -\sqrt{\xi^2+\eta^2}.$$

Examples.

1. Find the condition that the point (r, θ) may lie on the line (p, ω).
2. Show that the distance from the point (r, θ) to the line (p, ω) is $p - r\cos(\omega - \theta)$.

CHAPTER III

REPRESENTATION OF POINT AND LINE BY EQUATIONS

23. IN Chapter II (§ 17) it was shown that the point (x, y) lies on the line (ξ, η) if $\xi x + \eta y + 1 = 0$. For example, if the line be $(2, 3)$, the condition that (x, y) lie on it is $2x + 3y + 1 = 0$. This equation is satisfied by an indefinite number of pairs of values for x, y; *e.g.* by—

$$\begin{array}{llllllll} x = & -3, & -2, & -1, & 0, & +1, & +2, & +2\tfrac{1}{2}, \\ y = & \tfrac{5}{3}, & 1, & \tfrac{1}{3}, & -\tfrac{1}{3}, & -1, & -\tfrac{5}{3}, & -2. \end{array}$$

We may, in fact, give to one of the two coordinates any value we please, and then solve the equation for the remaining coordinate. This agrees with the geometrical fact that an indefinite number of points lie on any given line. If we think of a point as occupying in succession the various possible positions, that is, as moving along the line, the line is the locus of the point. The coordinates of the moving point satisfy the equation $2x + 3y + 1 = 0$, which is therefore called the equation of the line. Thus the equation of a line (p, q), that is, *the condition to which the coordinates of a point are subject if the point lies on the line* (p, q) is $px + qy + 1 = 0$.

Again, the condition of incidence of a line (ξ, η) and

a fixed point (h, k), namely, $h\xi + k\eta + 1 = 0$, is satisfied by an indefinite number of pairs of values for ξ, η; through the point (h, k) there pass an indefinite number of lines. We may think of one line as taking up in succession all these positions; the coordinates of the moving line are subject to the equation $h\xi + k\eta + 1 = 0$, which is called the equation of the point. *The equation of a point is the condition to which the coordinates of a line are subject if the line passes through the point.*

24. We have shown above that a line is represented by an equation of the first degree in point-coordinates (the point-equation of a line is of the first degree), and that a point is represented by an equation of the first degree in line-coordinates (the line-equation of a point is of the first degree). Conversely, an equation of the first degree in point-coordinates represents a line; and an equation of the first degree in line-coordinates represents a point. This must now be proved.

Let $ax + by + c = 0$, where $c \neq 0$, be the equation in point-coordinates; divide by c, to make the last term unity, and write p, q for $\frac{a}{c}, \frac{b}{c}$, thus reducing the equation to $px + qy + 1 = 0$. Now there *is* a line with coordinates p, q (for ξ, η may have any values), and this equation expresses that the point (x, y) lies on this line (p, q). Hence all points whose coordinates satisfy the equation $ax + by + c = 0$ lie on the line whose coordinates are $\frac{a}{c}, \frac{b}{c}$.

Again, let $a\xi + b\eta + c = 0$, where $c \neq 0$, be the equation

in line coordinates; this can be written in the form $h\xi + k\eta + 1 = 0 \left(\text{where } h = \dfrac{a}{c}, k = \dfrac{b}{c}\right)$, and thus written, it is seen to express that the line (ξ, η) passes through the point (h, k). Hence all lines whose coordinates satisfy the equation $a\xi + b\eta + c = 0$ pass through the point whose coordinates are $\dfrac{a}{c}, \dfrac{b}{c}$.

Example.—What conclusion is to be drawn if one of the two quantities a, b has the value zero?

25. Thus we have two ways of expressing the position of a point, (i) by its coordinates h, k, (ii) by its equation $h\xi + k\eta + 1 = 0$; and two ways of expressing the position of a line, (i) by its coordinates p, q, (ii) by its equation $px + qy + 1 = 0$.[1] The transition from the one to the other is immediate. The equation may, however, be in the form $ax + by + c = 0$, or $a\xi + b\eta + c = 0$, where $c \neq 1$; it must then be divided throughout by the quantity c, and thus reduced to the form $px + qy + 1 = 0$, or $h\xi + k\eta + 1 = 0$. For example, the coordinates of the line $4x - 3y - 5 = 0$ are $-\frac{4}{5}, +\frac{3}{5}$, for the equation is equivalent to $-\frac{4}{5}x + \frac{3}{5}y + 1 = 0$; the coordinates of the point $7\xi + 2\eta - 1 = 0$ are $-7, -2$, for the equation can be written—

$$-7\xi - 2\eta + 1 = 0.$$

Generally speaking, we use coordinates for one of the two elements, equations for the other; coordinates of points

[1] The second way expresses the position of the element in question *indirectly;* it leaves us to infer the position of the point from the information supplied as to *all lines through it,* and the position of the line from the information supplied as to *all points on it.*

with equations of lines, coordinates of lines with equations of points. But it is not necessary to restrict ourselves to this usage; we may when convenient represent a line sometimes by its equation, sometimes by its coordinates.

We have proved that unless $c = 0$, the equation $ax + by + c = 0$ represents a straight line whose coordinates are $\frac{a}{c}, \frac{b}{c}$. As c approaches the value zero, these coordinates become indefinitely great; and it was pointed out in § 13 that a line whose coordinates are both infinite passes through the origin. Thus $ax + by = 0$ represents a line through O, though it is not yet evident what particular line. (See § 26).

Note.—The term in an equation that contains neither x nor y is called the absolute term. If the absolute term in an equation of the first degree is zero, the line passes through the origin.

26. It can be shown without any use of line-coordinates that the geometrical statement, "a point P moves along a certain line," is exactly expressed by the algebraic statement, "the coordinates of P satisfy an equation of the first degree." To prove this, we consider separately lines parallel to an axis, or passing through the origin, and then lines not thus specialised in position.

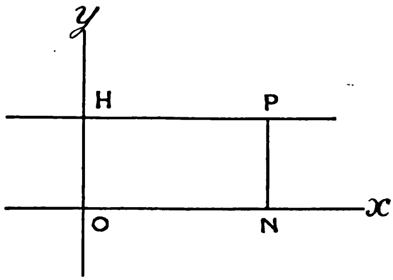


Fig. 27.

(i) A line parallel to the axis of x (Fig. 27). Let it meet Oy at H, where OH $= n$.

Then wherever P may lie on the line,

$$NP = OH = n,$$

$$\text{i.e.} \quad y = n.$$

Hence the equation of the line is $y = n$.

(ii) A line parallel to the axis of y (Fig. 28). Let it

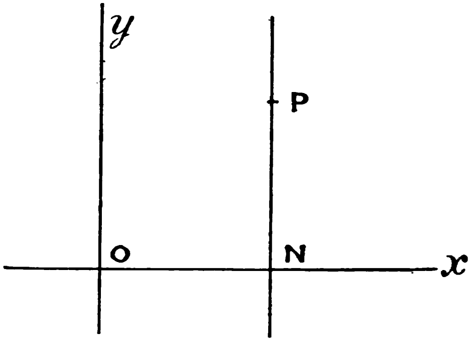


Fig. 28.

meet Oy at N, where ON $= h$. Then for all positions of P on the line, $x = h$. Hence the equation of the line is $x = h$.

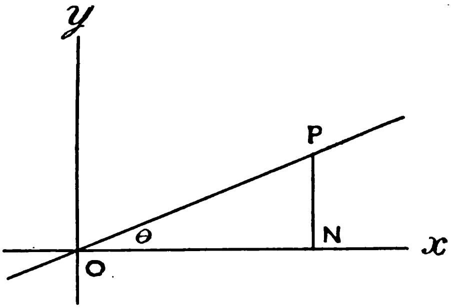


Fig. 29.

(iii) Let the line pass through O, and make with Ox e angle θ (Fig. 29.).

Then $$\frac{NP}{ON} = \tan\,\theta.$$

If m be written for $\tan\theta$, this becomes $\dfrac{y}{x} = m$. Hence the equation of the line is $y = mx$.

Definition. The angle θ is the *inclination* of the line; m ($= \tan\theta$) is the *slope* of the line.

(iv) A line in general does not pass through O, and is not parallel to either axis. Let the line meet Oy at H, where OH $= n$, and let the slope be m. Draw the line through O parallel to the given line (Fig. 30). The ordinate NP of any point on the given line exceeds the ordinate of Q by QP, which is equal to OH and therefore to n. We have therefore—

$$NP = NQ + n,$$

and $$NQ = m\,.\,ON.$$

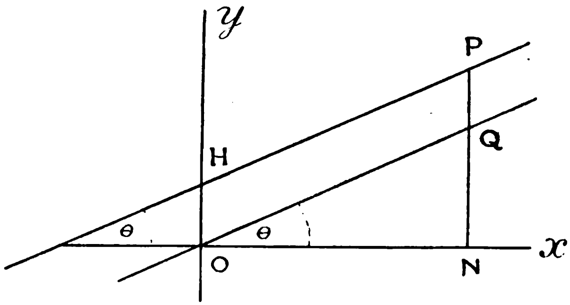


Fig. 30.

These give $$NP = m\,.\,ON + n,$$

i.e. $$y = mx + n.$$

Hence the equation of a line of slope m, making on the axis of y an intercept n, is $y = mx + n$.

Note.—Sometimes n is called the ordinate of the line at the origin.

Notice that the special cases $y = mx$ and $y = n$ are included under $y = mx + n$; they are obtained by writing $n = 0$, or $m = 0$. To reduce $y = mx + n$ to the remaining special form, $x = h$, we must have $\theta = \frac{\pi}{2}$, hence $m = \infty$, and also $n = \infty$; but the independent proof (given as (ii)) is to be preferred. In every case the resulting equation is seen to be of the first degree.

27. To complete this proof of the exact equivalence of the geometrical and algebraic statements (without the use of line-coordinates) it is necessary to prove the converse, namely, that an equation of the first degree represents a straight line. The equation $ax + by + c = 0$ can be written in the form $y = -\frac{a}{b}x - \frac{c}{b}$, unless $b = 0$. Now a line can be found to make with Ox any desired angle θ, and to make on Oy any desired intercept n. Since the tangent of an angle may have any value between $-\infty$ and $+\infty$, there does exist an angle whose tangent is $-\frac{a}{b}$; also we can certainly take on Oy an intercept $-\frac{c}{b}$. The line determined by this angle and this intercept has the equation $y = -\frac{a}{b}x - \frac{c}{b}$, which is the given equation. If

however $b = 0$, the given equation is $ax + c = 0$, that is, $x = -\dfrac{c}{a}$, which represents a line parallel to the axis of y. Thus in every case the equation of the first degree, $ax + by + c = 0$, represents a straight line.

28. In algebra it is shown that an equation may offer alternative solutions; *e.g.* the equation $x^2 - 6x + 8 = 0$, i.e. $(x - 2)(x - 4) = 0$, gives $x = 2$ or 4, either value satisfies the equation. Similarly an equation involving the coordinates of points or lines may offer alternative interpretations; e.g. $x^2 - y^2 = 0$, i.e. $(x - y)(x + y) = 0$, is satisfied by any point whose coordinates make either factor vanish, that is, by any point on the line $x - y = 0$, or by any point on the line $x + y = 0$. The locus of the given equation is composed of these two separate loci; it is said to *be reducible*, or to *break up*. Similarly—

$$(x + y)^2 - 3(x + y) + 2 = 0$$

breaks up into $(x + y - 1)(x + y - 2) = 0$;

the locus is composed of the two lines—

$$x + y - 1 = 0, \quad x + y - 2 = 0.$$

Examples.

Find the straight lines represented by the following equations—

1. $6x^2 - 7xy - 3y^2 = 0.$
2. $x^2 - 7xy + 10y^2 = 0.$
3. $x^2 - 4xy + 3y^2 = 0.$
4. $x^2 - 2xy + y^2 = 0.$
5. $(x - 1)^2 - 4(y - 3)^2 = 0.$
6. $(x + y)^2 - 6(x + y) + 8 = 0.$
7. $xy + 2x + 3y + 6 = 0.$

29. The fact that a line is represented by an equation of the first degree having once been established, we can find the equation of a line to satisfy given conditions.

Example i.—Find the line through the points (2, 3), (7, − 2).

Let the equation be $y = mx + n.$
The line passes through (2, 3), hence $3 = 2m + n,$
and through (7, − 2), hence $-2 = 7m + n.$

Solving these two equations for m and n we obtain—

$$m = -1, n = 5;$$

the equation of the line is therefore $y = -x + 5,$
that is, $x + y - 5 = 0.$

Example ii.—Find the line through (2, 3), (4, 6).

The equation is $y = mx + n,$
where
$$3 = 2m + n,$$
$$6 = 4m + n.$$

Hence $n = 0,\ m = \frac{3}{2}$; the equation of the line is $y = \frac{3}{2}x$, that is, $3x - 2y = 0.$

Example iii.—Find the line through (2, 3), (5, 3).

The equation is $y = mx + n,$
where
$$3 = 2m + n,$$
$$3 = 5m + n.$$

Hence $m = 0, n = 3$; the equation of the line is $y = 3$.

In this example it is at once clear from the given conditions that the line is parallel to the axis of x.

Example iv.—Find the line through (2, 3), (2, 4).

If the equation is $y = mx + n,$
we must have
$$3 = 2m + n,$$
$$4 = 2m + n.$$

These equations cannot be satisfied by finite values of m and n;

it is not possible for $2m + n$ to have the value 3 and also the value 4. It is clear however from the given conditions that the line is parallel to the axis of y, and is represented by $x = 2$ (see Fig. 31).

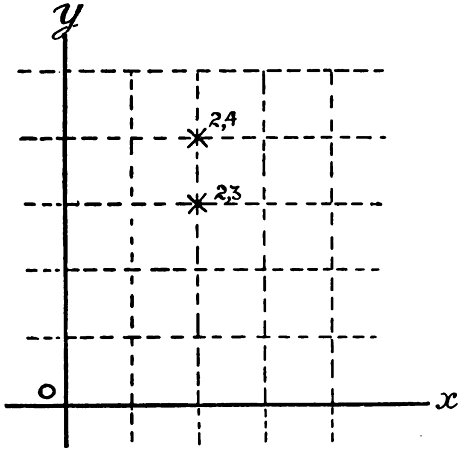


Fig. 31.

30. If the equation of a line be given as $ax + by + c = 0$, the direction is known, for we can find the slope at once by writing the equation in the form—

$$y = mx + n,$$

$$\text{i. e.} \quad y = -\frac{a}{b}x - \frac{c}{b}.$$

Hence
$$m = -\frac{a}{b}.$$

Example.—The line $x - y - 2 = 0$ gives for m the value 1; the line is inclined at an angle 45° to the positive direction of the axis

of x. The line $x + y - 2 = 0$ gives $m = -1$; the angle is $135°$ (Fig. 32).

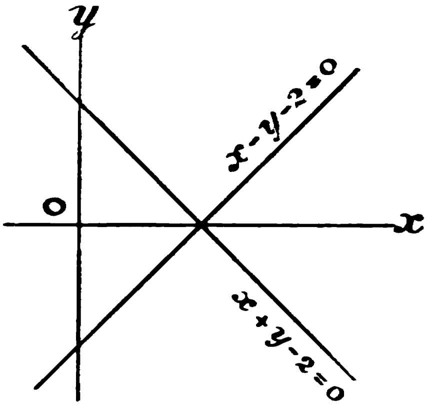


Fig. 32.

Examples.

1. Find the equations of the lines through the pairs of points $(3, 1), (7, 2)$; $(4, -3), (6, 1)$; $(-5, -4), (-7, -1)$.

2. Find the slope of each of the lines in Example 1.

3. Find the slopes of the lines $2x + y - 1 = 0$, $4x + 2y + 5 = 0$. Hence show that the lines are parallel.

4. Prove that the line through $(1, 10), (-1, -4)$ is parallel to the line through $(1, 2), (2, 9)$.

5. Prove that the lines $2x + y - 1 = 0, x - 2y + 3 = 0$ are perpendicular.

6. Show that $y = 3x + n$ is parallel to $y = 3x + 1$ for all values of n.

7. Show that $3x + 4y + k = 0$ is perpendicular to $4x - 3y = 0$ for all values of k.

8. Show that $ax + by + k = 0$ is parallel to $ax + by + c = 0$, and that $bx - ay + k = 0$ is perpendicular to $ax + by + c = 0$.

9. Find the equations of the lines that join the points $(7, 2), (3, 6), (-1, 4)$ to the points of bisection of the opposite sides of the triangle.

10. Find the slope of the line whose coordinates are p, q.

31. Two straight lines meet at a point. The two lines are represented algebraically by their equations, and the point they have in common is a point whose coordinates satisfy both equations. Hence we find the point of intersection of the lines by treating their equations as simultaneous. We thus obtain one pair of values for x, y. For example, the lines $3x + y - 9 = 0$, $7x - 4y - 2 = 0$, meet at the point $x = 2$, $y = 3$.

EXAMPLES.

1. Find the vertices of the triangle whose sides are—

$$7x + 24y - 31 = 0,\ 15x + 20y - 35 = 0,\ 4x + 3y - 14 = 0.$$

2. Find the point of intersection of—

$$y = mx + \frac{a}{m},\ y = m'x + \frac{a}{m'}.$$

3. Find the point of intersection of—

$$ax + by + c = 0,\ a'x + b'y + c' = 0.$$

32. If two lines are drawn from one point, they form an angle. But the lines we are concerned with are indefinite in extent; they do not terminate at their point of intersection, but extend beyond it. Thus they form four angles, which are equal in pairs; and unless the lines are perpendicular, when all four angles are equal, we can give two different answers to the inquiry, What is the magnitude of the angle made by the lines? Moreover, as explained in trigonometry, an angle can be described in a positive sense (counterclockwise), or a negative sense; in Fig. 33 the angle formed can be given with perfect accuracy as $+45°$, $-45°$, $+135°$, or $-135°$. (If we take into account angles greater than two right angles, there are infinitely many answers.) To obviate any doubt as

to what is meant, we must say more precisely what we understand by the angle. We regard the angle as traced out by a specified line of the two given lines, which starts

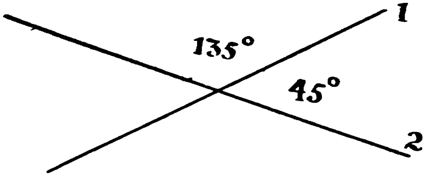


Fig. 33.

from coincidence with the other, and rotates in the positive sense about the point of intersection until it *first* takes the position assigned to it. In accordance with this definition, the angle made by 1 with 2 in Fig. 33 is 45°; the angle made by 2 with 1 is 135°. (With equal clearness we could speak of these angles as – 135° and – 45°, but it is more convenient to keep to the one form.) The angle thus defined, whether made by 1 with 2, or by 2 with 1, is *positive* and *less than two right angles.*

If, however, the lines are directed lines, the rotation must bring the positive direction on one to coincide with the

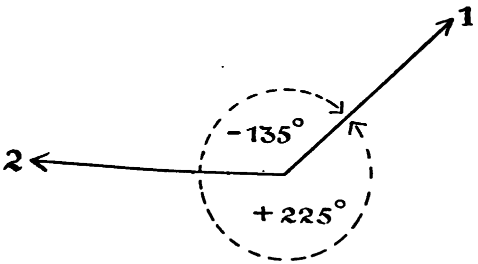


Fig. 34.

positive direction on the other. In Fig. 34, 2 makes with

1 an angle 135°; 1 makes with 2 an angle $-135°$, or $+225°$. We shall not have much occasion to consider the rotation of directed lines.

The inclination of a line, defined in § 26, is the angle that the line makes with the axis of x.

33. The angle that one line (not directed) makes with another is found from the slopes of the two lines. Let ϕ be the angle that $y = mx + n$ makes with $y = m'x + n'$. Then $\phi = \theta - \theta'$. (In Fig. 35, the lines are drawn so that θ is greater than θ'; hence ϕ is positive.)

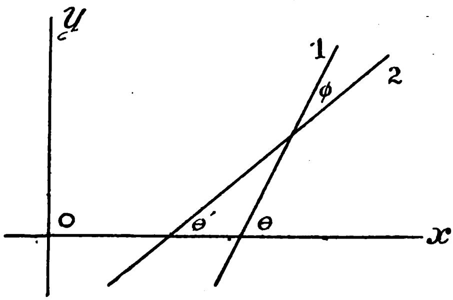


Fig. 35.

Hence
$$\tan\phi = \tan(\theta - \theta')$$
$$= \frac{\tan\theta - \tan\theta'}{1 + \tan\theta\tan\theta'}$$
$$= \frac{m - m'}{1 + mm'}.$$

If the lines are $ax + by + c = 0$ . . . . (1)

$a'x + b'y + c' = 0$ . . . . (2),

then $m = -\dfrac{a}{b}$, $m' = -\dfrac{a'}{b'}$; the angle that 1 makes with 2 is ϕ, where

$$\tan\phi = \frac{-\dfrac{a}{b} + \dfrac{a'}{b'}}{1 + \dfrac{aa'}{bb'}} = \frac{-ab' + a'b}{aa' + bb'}.$$

If the sine or cosine of the angle is required, it can be found at once from the value of the tangent. By the definition adopted, the angle is positive and less than π; hence whether the tangent is positive $\left(0 < \phi < \dfrac{\pi}{2}\right)$ or negative $\left(\dfrac{\pi}{2} < \phi < \pi\right)$, the sine is positive, while the cosine has the same sign as the tangent. For example, let $\tan\phi = \frac{3}{4}$; then since the hypothenuse of a right-angled triangle whose sides are 3, 4 is 5, we have $\sin\phi = \frac{3}{5}$, $\cos\phi = \frac{4}{5}$. Again, let $\tan\phi = -\frac{5}{12}$; the hypothenuse of a right-angled triangle with sides 5, 12 is 13, hence

$$\sin\phi = \tfrac{5}{13},\ \cos\phi = -\tfrac{12}{13}.$$

Examples.

1. Prove that the tangent of the angle made by the line (p, q) with the line (p', q') is $\dfrac{-pq' + p'q}{pp' + qq'}$.

2. Find the tangent of the angle made by each of the lines $6x + 7y + 10 = 0$, $x - 2y - 3 = 0$, $2x + y - 6 = 0$, with $3x + y - 4 = 0$.

3. Show that the lines $y - 4 = 0$, $x\sqrt{3} - y + 2 = 0$, $x\sqrt{3} + y + 3 = 0$ form an equilateral triangle. Find the vertices of the triangle. Find also the altitude and the area.

4. Show that the lines $3x + y + 4 = 0$, $3x + 4y - 15 = 0$, $24x - 7y - 3 = 0$ form an isosceles triangle. Find the altitude of the triangle.

34. From the value found for $\tan\phi$, we learn that two lines, $y = mx + n$, $y = m'x + n'$, are parallel if $m = m'$; this is obvious also from the meaning of m, m', since parallel lines have the same slope. The lines $ax + by + c = 0$, $a'x + b'y + c' = 0$ are parallel if $\frac{a}{b} = \frac{a'}{b'}$, which may be written $\frac{a}{a'} = \frac{b}{b'}$. The coordinates of parallel lines (p, q), (p', q') are proportional, namely, $\frac{p}{q} = \frac{p'}{q'}$, for the slope of a line p, q is $-\frac{p}{q}$.

Two lines are perpendicular if $\tan\phi = \infty$; that is, if the denominator of the value found for $\tan\phi$ vanishes. The condition of perpendicularity is therefore—

for (1) $\quad y = mx + n,\ y = m'x + n',\ mm' + 1 = 0$;
for (2) $\quad ax + by + c = 0,\ a'x + b'y + c' = 0,\ aa' + bb' = 0$;
and for (3) the lines $\quad (p, q), (p', q'), pp' + qq' = 0$.

From (1) we see that if the slope of a line is m, the slope of a perpendicular line is $-\frac{1}{m}$.

35. Any line parallel to a given line $y = mx + n$ must have the slope m; the equation is therefore $y = mx + k$, where different values for k give different lines, all in the same direction. Thus the equation $y = mx + k$ represents *all* lines parallel to $y = mx + n$; any one line of the system can be found by giving to k the appropriate value. In

particular, if the line is to pass through a given point (x', y'), we must have $y' = mx' + k$, hence $k = y' - mx'$, and the equation of the line is therefore—

$$y = mx + y' - mx'.$$

It is simpler to write this as follows:—The equation of a line in the given direction is—

$$y = mx + k,$$

this goes through (x', y') if

$$y' = mx' + k,$$

hence, eliminating k by subtraction, we find for the equation—

$$y - y' = m(x - x').$$

If the line be required to be parallel to $ax + by + c = 0$, the slope is $-\frac{a}{b}$, hence the equation is $y = -\frac{a}{b}x + h$, that is, $ax + by - bh = 0$, where h, and therefore bh, may have any value. Write k for $-bh$, the result then becomes —all lines parallel to $ax + by + c = 0$ are given by—

$$ax + by + k = 0;$$

any one line of the system can be found by giving to k the appropriate value. If we wish for the particular line that shall pass through (x', y'), the value of k must satisfy $ax' + by' + k = 0$; hence on eliminating k by subtraction from

$$ax + by + k = 0,$$
$$ax' + by' + k = 0,$$

we obtain as the equation of the desired line—

$$a(x - x') + b(y - y') = 0.$$

36. Any line perpendicular to $y = mx + n$ has the slope $-\frac{1}{m}$, and therefore the equation is $y = -\frac{1}{m}x + k$. This represents all lines perpendicular to $y = mx + n$; any particular line desired can be found by giving to k the proper value. If, for instance, the line is to pass through (x', y'), the condition to be satisfied is $y' = -\frac{1}{m}x' + k$. As before, we may substitute, in the equation already found for lines in the desired direction, the value of k given by this condition; but the simplest procedure is to subtract, and we then obtain—

$$y - y' = -\frac{1}{m}(x - x').$$

Similarly, since the slope of $ax + by + c = 0$ is $-\frac{a}{b}$, and the slope of any perpendicular line is therefore $+\frac{b}{a}$, we obtain $y = \frac{b}{a}x + h$, that is, $bx - ay + ah = 0$ (or $\frac{x}{a} - \frac{y}{b} + \frac{h}{b} = 0$) as the equation of any perpendicular line. Since h, and therefore ah and $\frac{h}{b}$, may have any value whatever, this result is better stated in the form—all lines perpendicular to $ax + by + c = 0$ are given by $bx - ay + k = 0$, or by the equivalent equation $\frac{x}{a} - \frac{y}{b} + k = 0$. If we wish for the particular line that shall pass through (x', y'), we find, by the same process as in the case of parallel lines, that the equation is—

$$b(x - x') - a(y - y') = 0,$$

which can also be written in the form—

$$\frac{x - x'}{a} = \frac{y - y'}{b}.$$

Notice that the coefficients of x, y in the equation of any line perpendicular to $ax + by + c = 0$ are found by either of the two rules—(1) invert the coefficients of x, y, and alter the sign of one, (2) interchange the coefficients of x, y, and alter the sign of one, but remember that the absolute term may have any value whatever; the absolute term in the given equation, namely c, does not appear in this equation. The direction of the given line is all that concerns us, and this does not in any way depend on c.

37. If two lines are given by one equation of the second degree (§ 28), the angle that they form may be found by breaking up the equation into two of the first degree, and then applying the formula for $\tan \phi$. But it is more convenient to have at hand a formula that shall obviate the necessity for breaking up the quadratic expression into its linear factors. If the lines are $y = mx + n$, $y = m'x + n'$, their combined equation is

$$(mx - y + n)(m'x - y + n') = 0;$$

that is,

$$mm'x^2 - (m + m')xy + y^2 + (mn' + m'n)x - (n + n')y + nn' = 0;$$

hence if the equation is given as—

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

comparison of coefficients shows that—

$$mm' = \frac{a}{b}, m + m' = -\frac{2h}{b}.$$

From these $(m - m')^2 = (m + m')^2 - 4mm'$

$$= \frac{4(h^2 - ab)}{b^2},$$

hence $$m - m' = \pm \frac{2\sqrt{h^2 - ab}}{b};$$

and consequently $$\tan\phi = \frac{m - m'}{1 + mm'}, = \pm \frac{\frac{2\sqrt{h^2 - ab}}{b}}{1 + \frac{a}{b}}$$

$$= \pm \frac{2\sqrt{h^2 - ab}}{a + b}.$$

The ambiguity is unavoidable; the two lines are given together, hence we cannot distinguish them as (1) and (2) without breaking up their equation. We are obliged therefore to use the ambiguous phrase, *the angle between the lines* (§ 32). If ϕ is the angle between two lines given by an equation of the second degree—

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

then $$\tan\phi = \pm \frac{2\sqrt{h^2 - ab}}{a + b}.$$

If $h^2 - ab$ is negative, the lines are imaginary.

The lines are parallel if $\tan\phi = 0$, that is, if $h^2 - ab = 0$; they are perpendicular if $\tan\phi = \infty$, that is, if $a + b = 0$.

38. In the investigations of § 36, the slope of the required line was given. Whether the slope be already known or not, the equation of a line through a given point (x_1, y_1) is—

$$y - y_1 = m(x - x_1), \text{ or } a(x - x_1) + b(y - y_1) = 0.$$

For exactly as above, the line

$$y = mx + n \quad (\text{or } ax + by + c = 0)$$

will pass through (x_1, y_1) if

$$y_1 = mx_1 + n \ (\text{or } ax_1 + by_1 + c = 0).$$

Subtracting, we obtain the equation in the form—

$$y - y_1 = m(x - x_1) \ (\text{or } a(x - x_1) + b(y - y_1) = 0),$$

where m (or $\frac{-a}{b}$) is to be determined by any other fact stated about the line. If the line is to pass through a second given point (x_2, y_2), we must have—

$$y_2 - y_1 = m(x_2 - x_1).$$

This shows that the slope, m, of the line through (x_1, y_1), (x_2, y_2) is $\frac{y_2 - y_1}{x_2 - x_1}$, or, which is the same thing, $\frac{y_1 - y_2}{x_1 - x_2}$. This value of m is to be used in the equation—

$$y - y_1 = m(x - x_1);$$

precisely the same result is obtained, and more conveniently, if we divide the two sides of the equations—

$$y - y_1 = m(x - x_1),$$
$$y_2 - y_1 = m(x_2 - x_1);$$

we thus obtain $$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1},$$

that is, $$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2},$$

as the equation of the line through (x_1, y_1), (x_2, y_2).

If a line passes through O, and has the slope m, its equation has already been found, $y = mx$ (§ 26). If the

line passes through O and through a point (x_1, y_1), we have the condition $y_1 = mx_1$; hence the equation becomes $\frac{y}{y_1} = \frac{x}{x_1}$, which also can be found at once from a figure

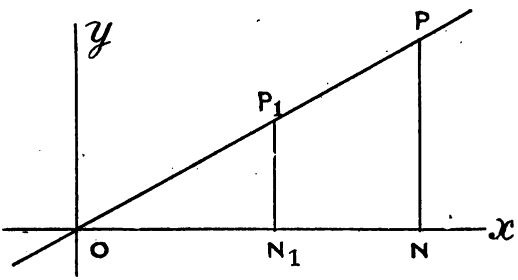


Fig. 36.

(Fig. 36), for
$$\frac{NP}{N_1P_1} = \frac{ON}{ON_1},$$
hence
$$\frac{y}{y_1} = \frac{x}{x_1}.$$

EXAMPLES.

1. Form the equations of the lines through the points (1, 2), (6, 4), (4, − 1) perpendicular to the opposite sides of the triangle formed by the three points.

2. Form the equations of the lines joining the points of bisection of the sides of the triangle in Example 1. Show from the equations that these lines are parallel to the sides of the triangle.

3. Do Examples 1 and 2, using the points (− 3, 3), (− 2, − 2), (2, − 1).

4. Write down the equation of the line through (x_1, y_1), perpendicular to $yy_1 = 2p(x + x_1)$.

5. Write down the equation of the line through (x_1, y_1), perpendicular to
$$\frac{xx_1}{\alpha} + \frac{yy_1}{\beta} = 1.$$

39. We have now found the equation of a straight line determined in various ways—

(1) If the coordinates are ξ, η, the equation of the line is $\xi x + \eta y + 1 = 0$. This is called the *dualistic form* of the equation of the line.

(2) If the intercepts on the axes are a, b, since $\xi = -\dfrac{1}{a}$ and $\eta = -\dfrac{1}{b}$, the equation of the line is $\dfrac{x}{a} + \dfrac{y}{b} = 1$, sometimes called the *intercept form* of the equation.

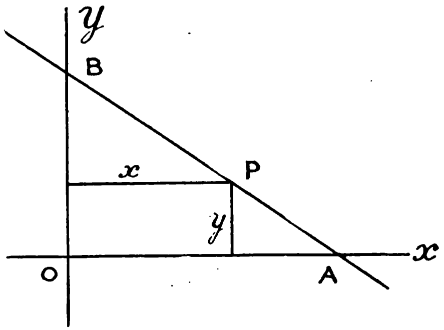


Fig. 37.

Note.—The independent proof of this is given by Fig. 37—

$$\begin{aligned} OA &= a,\ OB = b\,; \\ bx + ay &= 2\triangle OAP + 2\triangle OPB \\ &= 2\triangle OAB = ab, \\ \therefore \frac{x}{a} + \frac{y}{b} &= 1. \end{aligned}$$

(3) If the line has slope m, and passes through (x_1, y_1),

the equation is $y - y_1 = m(x - x_1)$, the *tangent form* of the equation.

In particular, if the line through (x_1, y_1) is parallel to $y = mx + n$, the equation is—

$$y - y_1 = m(x - x_1);$$

if it is parallel to $ax + by + c = 0$, the equation is—

$$a(x - x_1) + b(y - y_1) = 0;$$

if it is perpendicular to $y = mx + n$, the equation is—

$$y - y_1 = -\frac{1}{m}(x - x_1);$$

if it is perpendicular to $ax + by + c = 0$, the equation is—

$$b(x - x_1) = a(y - y_1),$$

or

$$\frac{x - x_1}{a} = \frac{y - y_1}{b}.$$

(4) If the line passes through (x_1, y_1), (x_2, y_2), the equation is—

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2};$$

which may be written in the form—

$$y = \frac{y_1 - y_2}{x_1 - x_2}x + \frac{x_1y_2 - x_2y_1}{x_1 - x_2}.$$

40. One other form must be given. The line may be determined by means of its distance from O, this being

given as of length p, making with OX an angle ω (see § 22). Then we have (Fig. 38)—

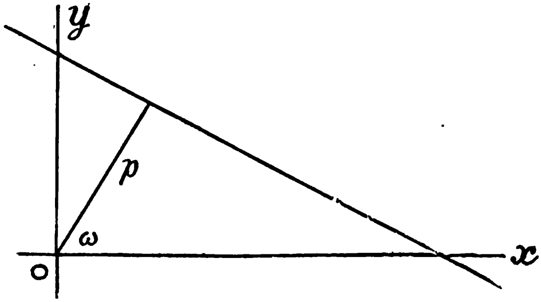


Fig. 38.

$$a \cos \omega = p, \; b \sin \omega = p,$$

$$\therefore \xi = -\frac{1}{p} \cos \omega, \; \eta = -\frac{1}{p} \sin \omega;$$

hence the equation $$\xi x + \eta y + 1 = 0$$

becomes $$-\frac{1}{p} \cos \omega \,.\, x - \frac{1}{p} \sin \omega \,.\, y + 1 = 0,$$

that is, $$x \cos \omega + y \sin \omega - p = 0.$$

This is called the *standard form* of the equation of the line.

Notice that in the standard form the last term is $-p$; since p is an essentially positive quantity (convention of § 14), this last term is always *negative.* Notice also that the sum of the squares of the first two coefficients is equal to unity.

41. Any one of these forms can be reduced to any other.

Example.—Reduce $5x - 12y + 10 = 0$ to the standard form.

The equation may be written—

$$5k.x - 12k.y + 10k = 0.$$

We are to make
$$5k = \cos\omega,$$
$$-12k = \sin\omega;$$
$$10k = -p.$$

The two equations that involve ω give, since $\cos^2\omega + \sin^2\omega = 1$,

$$25k^2 + 144k^2 = 1,$$
$$i.e.\quad 169k^2 = 1,$$
$$\therefore k = \pm\frac{1}{13}.$$

The equation $10k = -p$ shows that k must be negative; hence the proper value for k is $-\frac{1}{13}$, and the equation written in the standard form is—

$$-\frac{5}{13}x + \frac{12}{13}y - \frac{10}{13} = 0.$$

The process is perfectly general. The equation—

$$ax + by + c = 0$$

can be written $ka.x + kb.y + kc = 0$.

Here k must be determined so that (1) $ka = \cos\omega$, $kb = \sin\omega$; (2) kc is negative,

hence
$$k^2(a^2 + b^2) = 1,$$
$$\therefore k = \pm\frac{1}{\sqrt{a^2 + b^2}}.$$

The sign of k must be determined to satisfy (2), hence in any given example, the sign of k is determined without ambiguity.

The standard form of $ax + by + c = 0$ is therefore—

$$\frac{a}{\sqrt{a^2 + b^2}}x + \frac{b}{\sqrt{a^2 + b^2}}y + \frac{c}{\sqrt{a^2 + b^2}} = 0,$$

where the sign of the radical is assigned so that the last term is negative.

The equation $\xi x + \eta y + 1 = 0$, reduced to the standard form, is—

$$-\frac{\xi}{\sqrt{\xi^2 + \eta^2}}x - \frac{\eta}{\sqrt{\xi^2 + \eta^2}}y - \frac{1}{\sqrt{\xi^2 + \eta^2}} = 0.$$

Examples.

1. Find the equations of the lines through the pairs of points (1, 1), (7, − 7); (2, 3), (14, 8); (− 5, 1), (2, − 6).
2. Write the equations (Ex. 1) in the dualistic form, and give the coordinates of each line.
3. Find the slope of each line, and give the equations in the tangent form.
4. Find the intercepts made by each line on the axes, and give the equations in the intercept form.
5. Reduce the equations to the standard form. Find the distance from the origin to each of the given lines.
6. Find the sine of each angle of the triangle formed by the three lines.

42. It was proved in § 16 that if $px + qy + 1$ is positive, the point (x, y) lies on the same side of the line (p, q) as the origin, and conversely. We may express this in the form—at all points on the origin side of the line (p, q) the expression $px + qy + 1$ has a positive value; or, inasmuch as we are using equations for lines, *at all points on the origin side of the line* $px + qy + 1 = 0$, *the expression* $px + qy + 1$ *is positive.* The complete statement is—the straight line $px + qy + 1 = 0$ divides the plane into two regions; in one region, determined by the origin, the expression $px + qy + 1$ has positive values, in the other region this expression has negative values.

If the equation of the line is given in the form $ax + by + c = 0$, the expression $ax + by + c$ is $px + qy + 1$ multiplied by c, which is a known numerical quantity with sign positive or negative. Thus in the two regions $ax + by + c$ has still opposite signs, only we can no longer assert that the positive values lie in the region that contains the origin. The correct statement is now—the straight line $ax + by + c = 0$ divides the plane into two regions, in one of which the expression $ax + by + c$ is positive, while in the other it is negative.

When we have occasion to refer frequently to an algebraic expression it saves time, and is advantageous for other reasons also, to denote it by a single letter. We shall therefore denote algebraic expressions at present by the letters u, v, w. Let the expression $ax + by + c$ be denoted by u, then the result obtained becomes—*the straight line $u = 0$ divides the plane into two regions, in one of which u is positive, while in the other u is negative.*

Definition.—The value of the expression u at any point is called the *power of the point* with respect to the line $u = 0$.

To indicate that the value of u is to be taken at some definite point P, we employ a suffix, P; thus if P be (2, 3), and u be $5x - y + 4$, then $u_P = 10 - 3 + 4 = 11$. If the point is (x', y') or (x'', y'') we generally write u' or u''; thus if u be $ax + by + c$, u' means $ax' + by' + c$. Similarly u_1, u_2 stand for $ax_1 + by_1 + c$, $ax_2 + by_2 + c$, the values of u, i. e. $ax + by + c$, at the points (x_1, y_1), (x_2, y_2).

A similar notation is employed for any algebraic expression, not necessarily of the first degree; *e.g.* if

$u = x^2 + 4y^2 - 36$, then $u_1 = x_1{}^2 + 4y_1{}^2 - 36$; if P is the point (7, 1), then $u_P = 49 + 4 - 36 = 17$.

In Fig. 39 the values of the expression $2x + y - 4$ are

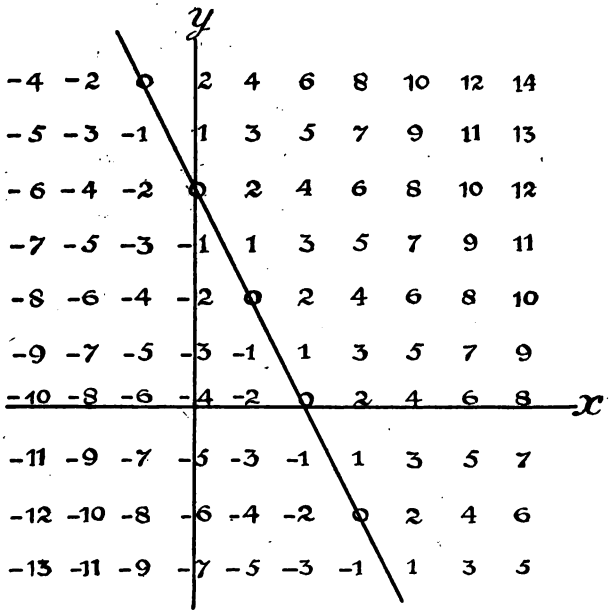


Fig. 39.

shown for integral values of x, y from $+6$ to -3; the line drawn is $2x + y - 4 = 0$. The values are of course found by simple calculation, and in the same way values of u can be inserted for intermediate (fractional) values of x, y.

Examples.

1. On a diagram mark the value of the expression $5x - 3y + 15$ for integral values of x, y from $x = +6$ to $x = -6$, and from $y = +6$ to $y = -6$.

2. Mark values of $x^2 + y^2 - 25$ from $x = +6$ to $x = -6$, $y = +6$ to $y = -6$.

3. Mark values of $x^2 - y$ from $x = +4$ to -4, $y = +10$ to -10.

4. Mark values of $x^3 - y$ from $x = +4$ to -4, $y = +10$ to -10.

43. The expression for the distance, d, from a point (x', y') to a line (ξ, η) was proved, in § 15, to be $\dfrac{\xi x' + \eta y' + 1}{\sqrt{\xi^2 + \eta^2}}$; hence the distance from a point (x', y') to a line $\xi x + \eta y + 1 = 0$ is $\dfrac{\xi x' + \eta y' + 1}{\sqrt{\xi^2 + \eta^2}}$. If the equation of the line be given in the form $ax + by + c = 0$, we have $\xi = \dfrac{a}{c}$, $\eta = \dfrac{b}{c}$; hence—

$$d = \frac{\frac{a}{c}x' + \frac{b}{c}y' + 1}{\sqrt{\frac{a^2}{c^2} + \frac{b^2}{c^2}}} = \frac{\frac{a}{c}x' + \frac{b}{c}y' + 1}{\sqrt{\frac{1}{c^2}}\sqrt{a^2 + b^2}}.$$

By the convention of § 14, the denominator is to be taken positive; hence if c is positive, this denominator is $\dfrac{1}{c}\sqrt{a^2 + b^2}$, while if c is negative, it is $-\dfrac{1}{c}\sqrt{a^2 + b^2}$, which has a positive value. To get rid of fractions, we multiply both numerator and denominator by c, and thus obtain the result that the distance from a point (x', y') to a line $ax + by + c = 0$ is $\dfrac{ax' + by' + c}{\sqrt{a^2 + b^2}}$, if c is positive; it is $-\dfrac{ax' + by' + c}{\sqrt{a^2 + b^2}}$ if c is negative. This can be stated in the form—the distance from x', y' to $ax + by + c = 0$ is

$\pm \dfrac{ax' + by' + c}{\sqrt{a^2 + b^2}}$, where the sign is to be determined so as to give a positive value if (x', y') is taken at the origin. (Remember the convention that the distance from the origin to a line is to be positive. § 14.)

Notice that the numerator is the value of u at the point (x', y'). Thus the distance from (x', y') to the line $u = 0$ (where $u = ax + ly + c$) is $\pm \dfrac{u'}{\sqrt{a^2 + b^2}}$. This shows that the value of u at any point, that is, the power of any point with respect to a line, is proportional to the distance from the point to the line. This fact may be illustrated by means of Fig. 39.

If the equation of the line is in the standard form, $x \cos \omega + y \sin \omega - p = 0$, we have $a^2 + b^2 = 1$. Hence the expression for the distance from (x', y') becomes $\pm (x' \cos \omega + y' \sin \omega - p)$, which in accordance with the convention of § 14 must be written—

$$d = -(x' \cos \omega + y' \sin \omega - p),$$

to give a positive result for the distance from the origin.

EXAMPLES.

1. The distances from a certain point to the lines $7x + 24y - 20 = 0$, $15x + 20y + 4 = 0$, $20x - 15y + 2 = 0$, are in the ratio $-2 : 6 : 3$. Find the point.

2. Find the vertices of the triangle whose sides are $3x - 5y - 7 = 0$, $8x + 7y - 39 = 0$, $11x + 2y + 15 = 0$. Show that each vertex is on the same side of the opposite line as the origin; hence that the origin is inside the triangle.

3. Find every point whose distances from the three lines in Ex. 1 are numerically equal. Which of these points is inside the triangle?

4. A point lies on the origin side of each of the lines

$7x + 24y - 5 = 0$, $20x - 15y - 31 = 0$, and its distances from the two lines are equal. Prove that the coordinates of the point satisfy an equation of the first degree, which represents a straight line through the point of intersection of the two lines. Show that the same result is obtained if the point and the origin are on opposite sides of each of the lines. Explain by means of a diagram.

5. A point and the origin are on the same side of one of the lines in Ex. 4, but on opposite sides of the other line; the distances of the point from the lines are numerically equal. Find and interpret the condition to which the coordinates of the point are subject, and illustrate by means of a diagram.

6. Find the locus of a point which moves so that its distances from the lines $7x + y - 2 = 0$, $x + y + 6 = 0$ are numerically equal and (i) alike in sign, (ii) opposite in sign. Represent the results on a diagram.

7. Find the equations of the two bisectors of the angles formed by $3x + 4y + 10 = 0$, $5x - 12y - 39 = 0$, by expressing that the distances to the two given lines from any point on either bisector are numerically equal, and alike in sign for one bisector, opposite in sign for the other bisector.

8. Find the equations of the two bisectors of the angles formed by the lines $ax + by + c = 0$, $a'x + b'y + c' = 0$.

9. A line passes through the mean centre of a number of points. Prove that the algebraic sum of its distances from the points is zero.

10. The distance from a point to the line $7x - y + 3 = 0$ is five times the distance to the line $x + y - 1 = 0$. Find the locus of the point.

11. Find the locus of a point which moves so that its distances from the lines $ax + by + c = 0$, $a'x + b'y + c' = 0$ are in a given ratio k.

12. A point, initially at (7, 2) moves so that its distances from the lines $3x - 4y + 1 = 0$, $8x + 6y - 3 = 0$ are in a constant ratio. Find the locus of the point. What is the ratio of the two distances?

44. It was proved in § 21 that the coordinates of any point on the line joining (x_1, y_1) to (x_2, y_2) can be written in the form—

$$x = \frac{x_1 + rx_2}{1 + r}, \quad y = \frac{y + ry_2}{1 + r},$$

where r = AP : PB. These are equivalent to—

$$x = x_1 + \frac{r}{1+r}(x_2 - x_1),$$

$$y = y_1 + \frac{r}{1+r}(y_2 - y_1).$$

If we write a single letter t for $\frac{r}{1+r}$, so that t = AP : AB (Fig. 40), these become—

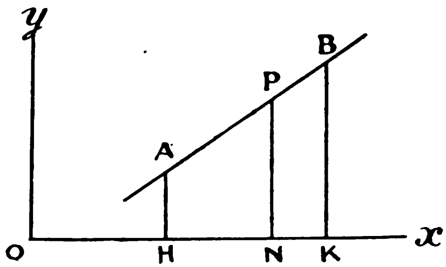


Fig. 40.

$$x = x_1 + t(x_2 - x_1)$$
$$y = y_1 + t(y_2 - y_1),$$

or, if we prefer,

$$x = x_1 + ht$$
$$y = y_1 + kt,$$

where h, k are written for $x_2 - x_1, y_2 - y_1$.

For any value of t, these expressions give the coordinates of a point on the line through (x_1, y_1), $(x_1 + h, y_1 + k)$; and all points on the line are obtained by using all values of t. Thus the coordinates of a point that moves along a line (*describes* a line) can be expressed in terms of a

single variable quantity, or *parameter*, t, in the form $x = a + bt$, $y = c + dt$.

This is not the only form of parametric expression for the coordinates of points on a straight line, though it is the simplest. We have already found

$$x = \frac{x_1 + rx_2}{1 + r}, \quad y = \frac{y_1 + ry_2}{1 + r},$$

and if we write for r, $\frac{gt}{f}$, these become—

$$x = \frac{fx_1 + gx_2 . t}{f + gt}, \quad y = \frac{fy_1 + gy_2 . t}{f + gt},$$

$$\text{i.e.} \quad x = \frac{a + bt}{f + gt}, \qquad y = \frac{c + dt}{f + gt}.$$

To obtain a relation between x, y that shall hold for all the points (that is, to obtain the equation of the line), we must eliminate the parameter t.

For example, let $x = 2 + 3t$, $y = 5 + 7t$. These give—

$$t = \frac{x - 2}{3}; \text{ also } t = \frac{y - 5}{7}.$$

Hence

$$\frac{x - 2}{3} = \frac{y - 5}{7},$$

$$i.e. \quad 7x - 3y + 1 = 0.$$

Examples.

1. Find the equation of the straight line described by—

$$\text{(i) } x = 4 - t, \quad y = 6 + 5t,$$
$$\text{(ii) } x = 4 + 9t, y = -1 + 2t.$$

2. Express in terms of a parameter the coordinates of any point on—

$$\text{(i) } 4x + y + 3 = 0,$$
$$\text{(ii) } 2x - 5y + 7 = 0.$$

3. Find the coordinates of the line described by—

$$x = a + bt, y = c + dt.$$

45. The equation of a line through the point of intersection of two given lines can be found by obtaining the coordinates of the point, and then applying the formula for a line through (x_1, y_1), namely, $y - y_1 = m(x - x_1)$. Unless however we require to know the coordinates of the point of intersection for some other purpose, the following process is better.

The equation $ax + by + c + \lambda(a'x + b'y + c') = 0$, being of the first degree, represents a straight line for any value of λ; and this straight line passes through the point P common to the lines $ax + by + c = 0$, $a'x + b'y + c' = 0$. For since P lies on the first of these lines, its coordinates make $ax + by + c = 0$, and since P lies on the second line, its coordinates make $a'x + b'y + c' = 0$; hence these coordinates satisfy—

$$ax + by + c + \lambda(a'x + b'y + c') = 0,$$

that is, this line does pass through P. The value of λ must be chosen to satisfy the remaining condition given for the determination of the line.

Example.—Find the line through the point common to $5x - 4y + 8 = 0$, $x + 6y - 28 = 0$, that shall—

(i) pass through the origin,
(ii) be parallel to $3x + y + 1 = 0$,
(iii) be perpendicular to $5x + 2y + 11 = 0$.

A line through the intersection of the two given lines is—

$$5x - 4y + 8 + \lambda(x + 6y - 28) = 0,$$

that is, $$(5 + \lambda)x + (-4 + 6\lambda)y + 8 - 28\lambda = 0.$$

(i) This passes through O if $8 - 28\lambda = 0$,

$$\therefore \lambda = \frac{2}{7}.$$

The equation becomes—

$$5x - 4y + 8 + \frac{2}{7}(x + 6y - 28) = 0,$$

$$i.e. \quad 7(5x - 4y + 8) + 2(x + 6y - 28) = 0,$$

$$i.e. \quad 37x - 16y = 0.$$

(ii) The line $(5 + \lambda)x + (-4 + 6\lambda)y + (8 - 28\lambda) = 0$ is parallel to $3x + y + 1 = 0$

if
$$\frac{5 + \lambda}{3} = \frac{-4 + 6\lambda}{1},$$

$$\therefore\ 5 + \lambda = -12 + 18\lambda,\ 17\lambda = 17,\ \lambda = 1.$$

The line is therefore $6x + 2y - 20 = 0,$

$$i.e. \quad 3x + y - 10 = 0.$$

(iii) The line $(5 + \lambda)x + (-4 + 6\lambda)y + (8 - 28\lambda) = 0$ is perpendicular to $5x + 2y + 11 = 0$

if
$$5(5 + \lambda) + 2(-4 + 6\lambda) = 0.$$

$$\therefore\ 17\lambda + 17 = 0,\ \lambda = -1.$$

The line is therefore—

$$4x - 10y + 36 = 0,$$

$$i.e. \quad 2x - 5y + 18 = 0.$$

Examples.

1. Find the line through the intersection of—

$$20x - y - 30 = 0,\ 15x - 4y + 55 = 0,$$

and (i) through O,

(ii) parallel to $6x + y - 40 = 0$,

(iii) perpendicular to $7x - y + 5 = 0$.

2. Find the line through the intersection of—

$$7x + 6y + 11 = 0,\ 21x - 5y - 1 = 0,$$

and (i) through O,

(ii) parallel to the axis of x,

(iii) parallel to the axis of y.

3. Find the line through the intersection of—

$$11x - 4y - 20 = 0,\ 5x + 9y - 46 = 0,$$

and (i) through (7, 2),

(ii) through (0, 0),

(iii) through (5, 6).

46. If the expressions $ax + by + c$, $a'x + b'y + c'$ are represented by u, v (§ 42), the result of the last section takes the form—any line through the intersection of two lines $u = 0$, $v = 0$ is represented by the equation $u + \lambda v = 0$, where by means of the quantity λ we can satisfy any other condition imposed on the line.

Let R be any point on the line $u + \lambda v = 0$ (Fig. 41).

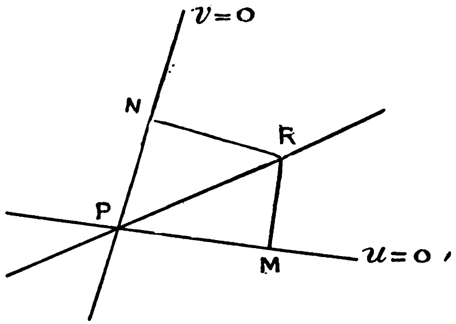


Fig. 41

The equation asserts that at the point R the values of the expressions u, v (namely, u_R, v_R) satisfy the relation $u_R + \lambda v_R = 0$. Now the values of u, v at the point R are known (§ 43) to be proportional to MR, NR; in fact, $u_R = \text{MR} \times \sqrt{a^2 + b^2}$, $v_R = \text{NR} \times \sqrt{a'^2 + b'^2}$. Hence the equation $u_R + \lambda v_R = 0$ shows that—

$$\text{MR}\sqrt{a^2 + b^2} + \lambda \text{NR}\sqrt{a'^2 + b'^2} = 0,$$

that is, MR : NR has a value, $-\dfrac{\lambda\sqrt{a'^2 + b'^2}}{\sqrt{a^2 + b^2}}$, which does not depend on the point R, but only on the two given lines

and the quantity λ. Hence the equation $u + \lambda v = 0$ represents the locus of a point R, which moves so that its distances from the two lines $u = 0$, $v = 0$ are in a constant ratio, a locus which is easily seen to be a straight line through P. For different lines this ratio has different values. (See Exs. 7, 8, 10, 11, 12, § 43, p. 67.)

47. Another consequence of the result of § 45 may be stated as follows:—If a straight line is partly determined, in such a manner that the coefficients in its equation contain a single undetermined quantity (or parameter) in the first degree, the line in all its positions passes through a fixed point. For if the coefficients a, b, c contain a single parameter λ in the first degree, that is, if

$$a = a_1 + \lambda a_2,\ b = b_1 + \lambda b_2,\ c = c_1 + \lambda c_2$$

(where the a_1, a_2, etc., are known quantities, any of which may be zero), the equation of the line is—

$$(a_1 + \lambda a_2)x + (b_1 + \lambda b_2)y + c_1 + \lambda c_2 = 0,$$

that is,

$$a_1x + b_1y + c_1 + \lambda(a_2x + b_2y + c_2) = 0,$$

the form already found for the equation of a line through the point common to—

$$a_1x + b_1y + c_1 = 0,\ a_2x + b_2y + c_2 = 0.$$

Example.—A line moves so that the sum of the reciprocals of the intercepts made on the axes is constant. Prove that the line in all its positions passes through a fixed point.

If the intercepts made on the axes are a, b, the given condition is—

$$\frac{1}{a} + \frac{1}{b} = \text{constant}$$
$$= k.$$

The equation of the line is $\frac{x}{a}+\frac{y}{b}=1$, where a and b vary from line to line.

Now $$\frac{1}{b}=k-\frac{1}{a},$$

hence the equation of the line is—

$$\frac{x}{a}+\left(k-\frac{1}{a}\right)y=1,$$

that is, $$\frac{1}{a}(x-y)+ky-1=0.$$

Write λ for $\frac{1}{a}$, then this equation becomes—

$$\lambda(x-y)+ky-1=0.$$

This involves the undetermined quantity λ in the first degree; the line, in all its positions, passes through the point common to $x-y=0$, $ky-1=0$, that is, through the point $\left(\frac{1}{k},\frac{1}{k}\right)$.

Notice the solution of this problem by line-coordinates.

Since $\xi=-\frac{1}{a}$, $\eta=-\frac{1}{b}$, the given condition $\frac{1}{a}+\frac{1}{b}=k$

becomes $$-\xi-\eta=k,$$

that is, $$\xi+\eta+k=0,$$

or $$\frac{1}{k}\cdot\xi+\frac{1}{k}\cdot\eta+1=0.$$

This equation, being of the first degree in ξ, η, represents a point (§ 24). Hence the line passes through a fixed point, whose coordinates are $\frac{1}{k}$, $\frac{1}{k}$.

48. If the coordinates of a line involve an undetermined quantity t in the first degree, the line in all its positions passes through a fixed point. For if the coordinates are $a+bt$, $c+dt$, the equation is—

$$(a+bt)x+(c+dt)y+1=0,$$

that is, $$ax+cy+1+t(bx+dy)=0,$$

and this, by § 45, passes through the fixed point common to the two lines—

$$ax + cy + 1 = 0,$$
$$bx + dy = 0.$$

Note.—The coordinates of the line—

$$a_1x + b_1y + c_1 + \lambda(a_2x + b_2y + c_2) = 0$$

are

$$\frac{a_1 + \lambda a_2}{c_1 + \lambda c_2}, \ \frac{b_1 + \lambda b_2}{c_1 + \lambda c_2}.$$

By the substitution of $\dfrac{c_1 t}{c_2(1-t)}$ for λ, these are reduced to—

$$\frac{a_1}{c_1} + t\left(\frac{a_2}{c_2} - \frac{a_1}{c_1}\right), \ \frac{b_1}{c_1} + t\left(\frac{b_2}{c_2} - \frac{b_1}{c_1}\right),$$

that is, to the form

$$\xi = p + qt \text{ (cf. § 44)},$$
$$\eta = r + st.$$

EXAMPLES.

1. Find the equation of the fixed point on the lines given by—

$$\xi = 7 + 2t,$$
$$\eta = 5 + 3t.$$

Find also the coordinates of the point.

2. Find the point through which pass the lines whose coordinates are—

$$\xi = 4 + 5t,$$
$$\eta = 3 - 2t.$$

3. A line passes between two fixed points (3, 4), (7, − 2) at equal distances from them. Show that in all its positions it passes through a certain fixed point.

CHAPTER IV

LOCI AND ENVELOPES

49. When points and lines have specified positions, either their coordinates are given directly (the point (3, 4), the line (6, 1), etc.), or sufficient facts are given to enable us to find the coordinates, which are thus given indirectly.

Example i.—A point is at equal distances from the points (8, 0), (4, − 2), and at a distance 5 from the origin. These two facts are expressed by the equations—

$$\sqrt{(x-8)^2+y^2} = \sqrt{(x-4)^2+(y+2)^2}, \quad . \quad . \quad . \quad . \quad (1)$$
$$\sqrt{x^2+y^2} = 5. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Before solving these equations we must rationalise; the first can then be reduced, and we have—

$$2x + y = 11, \; x^2 + y^2 = 25;$$
$$\therefore x^2 + (2x-11)^2 = 25,$$
$$5x^2 - 44x + 96 = 0,$$
$$(5x-24)(x-4) = 0.$$

Hence $\quad x = 4$, which gives $y = 3$,

or $\quad x = \dfrac{24}{5}$, which gives $y = \dfrac{7}{5}$;

the coordinates of the point are (4, 3) or $\left(\dfrac{24}{5}, \dfrac{7}{5}\right)$.

Example ii.—A line passes between the points (2, − 5), (1, − 6), at equal distances from them; also between and at equal distances from the points (3, − 2), (− 4, 3).

Let the line be (ξ, η). The two conditions are expressed by—

$$\frac{2\xi - 5\eta + 1}{\sqrt{\xi^2 + \eta}} = -\frac{\xi - 6\eta + 1}{\sqrt{\xi^2 + \eta^2}}, \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{3\xi - 2\eta + 1}{\sqrt{\xi^2 + \eta^2}} = -\frac{-4\xi + 3\eta + 1}{\sqrt{\xi^2 + \eta^2}}, \quad . \quad . \quad . \quad (2)$$

that is, by

$$3\xi - 11\eta + 2 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$-\xi + \eta + 2 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence $\xi = 3$, $\eta = 1$; the coordinates of the line are 3, 1, its equation is therefore $3x + y + 1 = 0$.

In these examples the algebraic expression of the given facts yields two algebraic equations to be satisfied by the coordinates of the point or line; we find the coordinates by solving these equations.

50. Again, the point or line may not have a definite position (either unique, or one of a group of alternative positions finite in number); it may be moveable. When a point is made to move, subject to some constraints, the path that it describes is called its locus. (The locus may also be defined as the aggregate of all the positions, indefinite in number, that the point is at liberty to occupy.) The constraint may be *actual;* for example, if the point is attached to a fixed point by a rotating rod of invariable length, the moving point describes a circle. Or it may be purely *formal;* that is, without actually constraining the point by physical means, we may state that the point is required to remain at a given distance from a certain fixed point. From the point of view of geometry, the result is the same in either case; the geometrical statement is that the distance of a point P from the fixed point C is constant,

the locus is therefore a circle. Expressed algebraically, the geometrical condition leads to an equation, the equation of the circle. If the centre be (p, q) and the constant distance (the radius of the circle) be r, the algebraic expression of the geometrical statement $\mathrm{CP} = r$ is $\sqrt{(x-p)^2 + (y-q)^2} = r$ (§ 12). When rationalised, this takes the form—

$$(x-p)^2 + (y-q)^2 = r^2.$$

For example, the equation of the circle with centre (3, 4) and radius 5 is—

$$(x-3)^2 + (y-4)^2 = 25,$$

that is,

$$x^2 + y^2 - 6x + 8y = 0.$$

51. If a point is free to move, not at random but subject to conditions, its coordinates will necessarily be subject to *one* equation. For if there were no condition imposed on x, y, these quantities could have any values, and thus the point could be anywhere in the plane, which contradicts the statement of the case. Now a condition imposed on x, y may be of various forms, for instance—

(i) Some expression that involves x, y has a definite value; this yields an equation $f(x, y) = k$, e. g. $x^2 + y^2 = a^2$.

(ii) Some expression that involves x, y is in a definite relation to some other expression; this reduces to (i), e.g. $f(x, y) = kf'(x, y)$ is $f(x, y) - kf'(x, y) = 0$, that is, $\mathrm{F}(x, y) = 0$; e. g. $\sin x = \tan y$; $e^{x+y} = x^2$; $(x-y)^2 = 3(x+y)^2$.

(iii) The relation between x, y may be implied, given indirectly, by means of expressed relations of x, y to an auxiliary quantity or parameter; *e. g.* a point moves so that the abscissa varies as the time, and the ordinate as the

square of the time, during which the point has moved. That is, if t be the time, $x = at$, $y = bt^2$. The elimination of the parameter t gives the relation between x, y, namely, $bx^2 - a^2y = 0$.

Thus whatever the form in which the condition is given, it can be written as $F(x, y) = 0$.

If the coordinates x, y were subject to *two* such relations $f(x, y) = 0$, $F(x, y) = 0$, the point would not be free to move. For the elimination of y from these equations would lead to an equation in x; hence x could not assume all values, but only those that satisfy this equation. Each value of x would be associated with certain values of y (in virtue of the given relations), and thus there would be scattered positions in the plane which the point could occupy in conformity with the conditions of the problem, but not a path along which it could move.

We see then that if a point is free to move, not arbitrarily but subject to some conditions (whether the necessary constraints can be realised physically or not), its coordinates are subject to a *single* equation. This is, the equation of the locus of the point.

52. To find the equation of the locus of a moving point—

(i) State the effect of the actual or formal constraints, that is, the conditions of the problem, in geometrical language.

(ii) Translate these statements, piece by piece, into algebraic language, denoting the coordinates of the moving point by (x, y), or (x', y').

(iii) Then combine and arrange these algebraic statements as suggested by the nature of the problem; this leads to the algebraic statement of the final result, but it may be necessary to rationalise and arrange this in order to find the equation of the locus in the most convenient form.

Example.—A point P moves so that the sum of its distances from the points S(5, 0) and S′(– 5, 0) is equal to 26; find the locus.

Let P be $(x, y.)$

(i) $$SP + S'P = 26.$$

(ii) By formula, $$SP = \sqrt{(x-5)^2+y^2},$$
$$S'P = \sqrt{(x+5)^2+y^2},$$
$$\therefore \sqrt{(x-5)^2+y^2} + \sqrt{(x+5)^2+y^2} = 26.$$

(iii) $$\therefore \sqrt{(x-5)^2+y^2} = -\sqrt{(x+5)^2+y^2} + 26,$$

squaring, $$(x-5)^2+y^2 = (x+5)^2+y^2 - 52\sqrt{(x+5)^2+y^2} + 676,$$

i. e. $$x^2-10x+25+y^2=x^2+10x+25+y^2+676-52\sqrt{(x+5)^2+y^2},$$
$$\therefore 52\sqrt{(x+5)^2+y^2} = 20x + 676,$$
i. e. $$13\sqrt{(x+5)^2+y^2} = 5x + 169;$$

squaring, $$169(x^2+10x+25+y^2) = 25x^2 + 1690x + 169^2,$$
$$\therefore 144x^2 + 169y^2 = 169^2 - 25 . 169$$
$$= 169 \times 144,$$

hence $\dfrac{x^2}{169} + \dfrac{y^2}{144} = 1$ is the equation of the locus.

53. By the same argument as in the case of a point it is shown that if a line can move in a plane, not arbitrarily but in conformity with some law, its coordinates are subject to a single equation. For example, if a line is

required to remain at a constant distance k from a given point (p, q), its coordinates ξ, η must satisfy—

$$\frac{p\xi + q\eta + 1}{\sqrt{\xi^2 + \eta^2}} = k,$$

that is, $k^2(\xi^2 + \eta^2) - (p\xi + q\eta + 1)^2 = 0$, a relation of the form $F(\xi, \eta) = 0$.

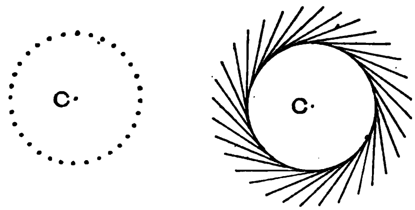


Fig. 42.

In Fig. 42, a number of lines, 32, are represented, all at the same distance from the point C; this figure shows that these lines indicate a curve, and quite as clearly as the same number of detached points. The more points or lines marked, the more plainly is the curve indicated. In this particular example it is evident to the eye that the curve is a circle; and the geometrical condition to which the line is subject is recognised as the condition that makes the line a tangent to a circle of centre (p, q) and radius k. The circle is called the envelope of the lines; the line equation of the circle is that found above.

The equation $F(\xi, \eta) = 0$, which is the algebraic expression of the constraints (actual or formal) to which a moving line is subject, is called the line equation of the envelope. The curve which is indicated by the lines is

called the envelope of the lines; in the example given it is seen that the lines are tangents to their envelope.[1]

In general, a curve found as a locus may be looked upon also as an envelope, the envelope of its tangents (§ 78); and a curve found as an envelope may be looked upon also as a locus, the locus of the points of contact of the tangents.

54. To find the equation of the envelope of a moving line—

(i) State the conditions of the problem in geometrical language.

(ii) Translate these statements, piece by piece, into algebraic language, denoting the coordinates of the moving line by (ξ, η) or (ξ', η').

(iii) Then combine and arrange these algebraic statements as suggested by the nature of the problem; it may be necessary to rationalise and arrange the result thus obtained in order to find the equation of the envelope in the most convenient form.

Example.—Two circles of radius $2a$, a move in the same direction along the axis of x. Their centres are initially at O, and the distance of the first centre from O is always one-half the distance of the second centre from O. Find the envelope of the exterior common tangents.

Let the centre C_1 of the first circle be at $(p, 0)$ at any time; then

---

[1] A system of points cannot *always* be described by a moving point, hence a system is not necessarily a locus; and a system of lines cannot *always* be described by a moving line, that is, it is not necessarily an envelope. But the systems of points and lines that will be met with in this book are of a simple type, expressible respectively as loci and envelopes.

the centre, C_2, of the second circle is at $(2p, 0)$. Let a common exterior tangent be (ξ, η).

(i) The distance from C_1 to $(\xi, \eta) = 2a$,
the distance from C_2 to $(\xi, \eta) = a$.

(ii) Since the tangent is exterior to the circles, the two centres and O are on the same side of the tangent. Hence—

$$\text{the distance from } C_1 \text{ to } (\xi, \eta) = \frac{p\xi + 1}{\sqrt{\xi^2 + \eta^2}},$$

$$\text{and the distance from } C_2 \text{ to } (\xi, \eta) = \frac{2p\xi + 1}{\sqrt{\xi^2 + \eta^2}};$$

the conditions, expressed algebraically, are therefore—

$$\frac{p\xi + 1}{\sqrt{\xi^2 + \eta^2}} = 2a, \quad \frac{2p\xi + 1}{\sqrt{\xi^2 + \eta^2}} = a.$$

(iii) In these p is a parameter, which must be eliminated to give the relation satisfied by ξ, η in all admissible positions of the circles. The equations may be written—

$$2a\sqrt{\xi^2 + \eta^2} = p\xi + 1,$$
$$a\sqrt{\xi^2 + \eta^2} = 2p\xi + 1,$$

hence, multiplying the first by 2 and subtracting the second, we find—

$$3a\sqrt{\xi^2 + \eta^2} = 1,$$

$$\therefore \sqrt{\xi^2 + \eta^2} = \frac{1}{3a}, \quad \xi^2 + \eta^2 = \frac{1}{9a^2}.$$

This is the line-equation of the envelope, and in finding this we have solved the problem algebraically. There still remains the interpretation of the result. This result may be written in the form—the line moves so that its distance from O (which is $\frac{1}{\sqrt{\xi^2 + \eta^2}}$, § 12) is $3a$, hence in all its positions it is a tangent to a circle, centre O, radius $3a$.

Example.—Find the envelope of the interior common tangents.

55. The interpretation of the algebraic conclusion depends on our recognition of the geometrical equivalent of the final algebraic statement, and while a very little

knowledge of analytical geometry may enable us to obtain an algebraic solution of some particular problem, the interpretation of the result may require a much more extensive acquaintance with the subject, involving a general knowledge of different curves. The problem is not completely solved, geometrically, unless the interpretation of the algebraic result is given. The two following examples illustrate the complete solution of two problems.

Example i.—Find the locus of a point which moves so that its distance from the point (4, 0) is twice its distance from the point (1, 0).

(i) State in geometrical terms—

$$AP = 2 . BP; \text{ where A is } (4, 0), \text{ B is } (1, 0).$$

(ii) Express in algebraic language—

$$\sqrt{(x-4)^2 + y^2} = 2\sqrt{(x-1)^2 + y^2}.$$

(iii) Rationalise and arrange—

$$(x-4)^2 + y^2 = 4[(x-1)^2 + y^2],$$

$$\text{i.e. } x^2 - 8x + 16 + y^2 = 4x^2 - 8x + 4 + 4y^2,$$

$$\therefore 3x^2 + 3y^2 = 12,$$

$$x^2 + y^2 = 4.$$

(iv) Interpret geometrically—hence the locus of the point is a circle, centre O, radius 2.

Example ii.—A line moves so that the sum of its distances from the points (4, 1), (– 4, – 1) is equal to 12. Find the envelope, if the line does not pass between A, B.

(i) State in geometrical terms—

$$AM + BN = 12 \text{ (Fig. 43).}$$

(ii) Express in algebraic language—

$$AM = \frac{4\xi + \eta + 1}{\sqrt{\xi^2 + \eta^2}}$$

$$BN = \frac{-4\xi - \eta + 1}{\sqrt{\xi^2 + \eta^2}},$$

hence $$\frac{4\xi + \eta + 1}{\sqrt{\xi^2 + \eta^2}} + \frac{-4\xi - \eta + 1}{\sqrt{\xi^2 + \eta^2}} = 12$$

$$\therefore \frac{2}{\sqrt{\xi^2 + \eta^2}} = 12.$$

(iii) Rationalise and arrange—

$$\xi^2 + \eta^2 = \frac{1}{36}.$$

(iv) Interpret geometrically—since $\dfrac{1}{\sqrt{\xi^2 + \eta^2}} = 6$, the envelope is a circle, centre O, radius 6.

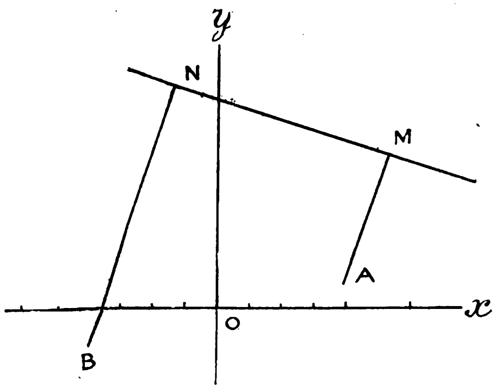


Fig. 43.

Examples.

Express in geometrical language the conditions to which the point P moving on a table is subjected by the following mechanical constraints (1—11).

1. The ends of a string are knotted together, and the loop thus formed is laid on the table, and slipped over two pegs fixed in the table; the loop is kept stretched by a moving point P.

2. A bar slides along a fixed line, to which it is perpendicular. A string of length equal to that of the bar has one end fastened to a fixed point of the table, and the other end fastened to the end of the bar that is not on the fixed line. The string is kept stretched by a moving point P, which slides along the bar.

3. The same constraints as in No. 2, only now the length of the string is less than that of the bar.

4. The same constraints as in No. 2, only now the length of the string is greater than that of the bar.

5, 6, 7. The same constraints as in 2, 3, 4, except that the bar makes with the fixed line an angle different from a right angle.

8, 9, 10. The same constraints as in 2, 3, 4, except that the bar, instead of sliding along a fixed line, rotates about a fixed point.

11. Three pegs S, C, S′ are equally spaced on a line; P is on the table, above the line SCS′. Two strings are attached to P, one passes below S and then round C, the other below S′ and then round C. The two strings are pulled away from the point occupied by P.

Find the equation to which the coordinates of the moving point or line are subject in virtue of the following geometrical conditions (12—35). Interpret the result where you are able.

12. The distances of a moving point from the two points $(a, 0)$, $(-a, 0)$ are equal.

13. The sum of the squares of the distances of a point from the two points $(a, 0)$, $(-a, 0)$ is constant, $= k^2$.

14. The sum of the squares of the distances of a point from n fixed points is constant, $= k^2$.

15. The ratio AP : BP is constant, $= k$, where A, B are two fixed points $(a, 0)$, $(-a, 0)$.

16. The product of the distances of a point from the two fixed points $(a, 0)$, $(-a, 0)$, is constant, $= k^2$.

17. The sum of the squares of the distances from a point to the two fixed lines (p, q), (p', q') is constant, $= k^2$.

18. The distance of a point from the fixed point $(f, 0)$ is equal to its distance from the axis of y.

19. The distance of a point from the fixed point (f, g) is equal to its distance from the line $lx + my + n = 0$.

20. The product of the distances of the point from the lines $x = a$, $x = -a$, is in a given ratio $(k:1)$ to the square of its distance from the axis of x.

21. The product of the distances of the point from the lines $x = a$, $x = -a$ is in a given ratio to the product of its distances from the lines $y = b$, $y = -b$.

22. The distance of a point from the fixed point $(ae, 0)$ is e times its distance from the line $x = \frac{a}{e}$. (i) $e < 1$, (ii) $e > 1$.

23. A point is inside a square of side $2a$, and the product of its distances from one pair of parallel sides is equal to the product of its distances from the other pair of sides. (The origin may be taken at the centre of the square, the axes of coordinates parallel to the sides.)

24. The same as 23, except that the point is in the region separated from the interior of the square by a single side.

25, 26. The same as 23, 24, only using a rectangle of sides $2a$, $2b$ instead of a square.

27. The sum of the distances of a line from a number of fixed points is constant, the origin being the mean centre of the fixed points, and all the points on the same side of the line.

28. The product of the distances of a line from two points $(4, 0)$, $(-4, 0)$ on the same side of the line is constant, $= 9$.

29. The product of the distances of a line from two points $(5, 0)$, $(-5, 0)$ on opposite sides of the line is constant, $= 16$.

30. A line of constant length k moves with its extremities on the axes.

31. The point bisects a line of constant length k whose extremities are on the axes.

32. A line makes with the axes a triangle of constant area k.

33. The point bisects a line which makes with the axes a triangle of constant area k.

34. A line moves so that the difference of the squares of its distances from the two fixed points $(a, 0)$, $(-a, 0)$ is constant, $= k^2$.

35. A line moves so that the sum of the squares of its distances from the two fixed points $(a, 0)$, $(-a, 0)$ is constant, $= k^2$.

56. Entirely distinct from the accurate geometrical interpretation of the result is the graphical interpretation. We may not recognise what curve is represented by the equation that we find as that of the locus of a moving point; but in any case we can construct a diagram that shall give a picture of the result, by finding a number of pairs of values of x, y that satisfy the equation. We find these values by giving to x (or y) a series of arbitrarily selected values, and solving the resulting equation for y (or x). We thus obtain a number of detached

points, which we can make as close together as we please, by selecting for x values close enough together; for instance, we may take for x the values 0, $\pm$ ·01, $\pm$ ·02, etc. These points indicate to the eye a curve, the *graph* of the equation, which cannot differ much from the true locus, inasmuch as this does pass through every one of the points found.

For example, the equation $y = x(x-1)(x-2)$ gives

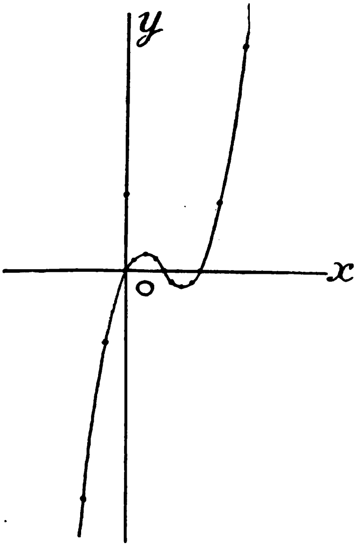


Fig. 44.

the following pairs of values, which indicate the curve, as shown in Fig. 44.

$$x = -1, -\tfrac{1}{2}, 0, \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{3}{4}, 1, 1\tfrac{1}{4}, 1\tfrac{1}{2}, 1\tfrac{3}{4}, 2, 2\tfrac{1}{2}, 3, \ldots.$$
$$y = -6, -\tfrac{15}{8}, 0, \tfrac{21}{64}, \tfrac{3}{8}, \tfrac{15}{64}, 0, -\tfrac{15}{64}, -\tfrac{3}{8}, -\tfrac{21}{4}, 0, \tfrac{15}{8}, 6. \ldots.$$

For the actual determination of the curve it would be necessary to ascertain what happens in the intervals between the points found, instead of assuming without proof that the curve passes simply and smoothly through these points. But this requires more elaborate processes than any at our disposal at this stage of the work. For the present we must content ourselves with the simple process given above, which is called "plotting the equation"; it does not give us accurate detailed knowledge of the curve, but it does give a good general idea of the appearance.

Although the proofs of analytical geometry are purely algebraic, with no dependence on any diagram, yet it is well to make a practice of representing conditions and results in a diagram. The diagram itself proves nothing; but it shows us what we have proved; and indeed in many cases it helps us to see precisely what it is that we wish to prove.

EXAMPLES.

Plot the following equations--

1. $y = x^2$.
2. $y = x^3$.
3. $y = x^2(x - 1)^2$.
4. $y = x^2(x^2 - 1)$.
5. $y^2 = x(x - 1)(x - 2)$. (Notice $y = \pm$.)
6. $xy = 1$.
7. $xy^2 = 1$.

57. In many problems that involve the finding of a locus or an envelope the conditions are most easily expressed in terms of auxiliary variable quantities (parameters); the elimination of these parameters leads to an

equation connecting the coordinates of the moving element (point or line). There may be only one parameter to eliminate; *e.g.* in an illustration already given (§ 51) $x = at$, $y = bt^2$, from which $\left(\frac{x}{a}\right)^2 = \frac{y}{b}$; or there may be more than one, connected by relations.

Example i.—Find the locus of the point half-way between the origin and a point that describes the curve $4x^2 + 9y^2 = 36$.

Let Q, a point on the curve, be (x', y'), then $4x'^2 + 9y'^2 = 36$.

The point P, half-way between O and Q, has coordinates—

$$x = \frac{x'}{2}, \quad y = \frac{y'}{2}.$$

From these three equations x', y' must be eliminated. Since $x' = 2x$ and $y' = 2y$, the result, obtained by the substitution of these values in $4x'^2 + 9y'^2 = 36$, is—

$$16x^2 + 36y^2 = 36,$$

that is,

$$4x^2 + 9y^2 = 9.$$

Example ii.—Find the locus of the vertex of a right angle whose sides pass through the points $(a, 0)$, $(-a, 0)$.

Let the two lines have slopes m, m'; their equations are then—

$$y = m(x - a),$$
$$y = m'(x + a).$$

By the condition of the problem, $mm' = -1$ (§ 34). From these three equations m, m' must be eliminated. The first two can be written—

$$m = \frac{y}{x - a}, \quad m' = \frac{y}{x + a};$$

the substitution of these expressions for m, m' in the equation gives

$$\frac{y^2}{x^2 - a^2} = -1,$$

that is,

$$x^2 + y^2 = a^2.$$

The locus is therefore a circle, whose centre is half-way between the two given points.

Examples.

1. Show that the locus of the vertex, if the angle is 45°, is $x^2 + y^2 \pm 2ay - a^2 = 0$. Explain the double sign in the result.

2. Two lines start from coincidence and rotate about $A(a, 0)$, $B(b, 0)$, the speed of rotation about A being twice as great as that about B. Find the locus of the common point when the rotations are (i) in the same direction, (ii) in opposite directions.

(i) $(x - a)^2 + y^2 = (a - b)^2$.
(ii) $3x^2 - y^2 - 2(a + 2b)x + 2ab + b^2 = 0$.

58. More generally, if Q (x', y') describes some curve, $f(x, y) = 0$, and the position of P (x, y) depends on that of Q so that x, y are expressed in terms of x', y' by equations $x = F_1(x', y')$, $y = F_2(x', y')$, the locus of P is found by eliminating x', y' from the three equations—

$$x = F_1(x', y'), \quad y = F_2(x', y'), \quad f(x', y') = 0,$$

the last of which expresses that the point (x', y') lies on the given curve $f(x, y) = 0$. When it is possible to solve two of the equations for x', y', the substitution in the remaining equation of the values thus found gives the desired result. This process gives, for example, the locus of the point *inverse* to Q with respect to a given circle, when Q describes any curve.

Definition.—Two points P, Q are inverse with respect to a circle of radius k when they lie on a diameter, at distances from the centre C connected by the relation $CP . CQ = k^2$.

Example.—Find the curve inverse to $x^2 - y^2 = a^2$ with respect to the circle $x^2 + y^2 = k^2$.

The centre of this circle is the origin; the radius is k. Hence the points $P(x, y)$, $Q(x', y')$ lie (1) on a line through O, (2) at distances which satisfy $OP . OQ = k^2$.

The condition (1) is expressed by—

$$x' = \lambda x,$$
$$y' = \lambda y;$$

the condition (2) is expressed by—

$$\sqrt{x^2 + y^2} \,.\, \sqrt{x'^2 + y'^2} = k^2.$$

Hence $$\sqrt{x^2 + y^2} \,.\, \lambda \sqrt{x^2 + y^2} = k^2,$$

$$\text{i.e.}\quad \lambda = \frac{k^2}{x^2 + y^2},$$

from which we find $$x' = \frac{k^2 x}{x^2 + y^2},$$

$$y' = \frac{k^2 y}{x^2 + y^2}.$$

Now Q describes the curve $x^2 - y^2 = a^2$;
hence $x'^2 - y'^2 = a^2$.

On substituting for x', y', in this equation, their expressions in terms of x, y, we find as the equation of the locus of P,

$$\frac{k^4 x^2}{(x^2 + y^2)^2} - \frac{k^4 y^2}{(x^2 + y^2)^2} = a^2,$$

that is, $$a^2(x^2 + y^2)^2 - k^4(x^2 - y^2) = 0.$$

EXAMPLES.

1. Plot the curve $x^2 - y^2 = 1$, and on the same diagram, the inverse of this with respect to the circle $x^2 + y^2 = 1$.

2. Do this also for $4x^2 + 9y^2 = 36$.

3. Find the inverse of the straight line $x = 1$ with respect to the circle $x^2 + y^2 = 1$. Also of the lines $x = a$, $x + y - 4 = 0$.

4. Prove that the inverse of a line with respect to a given circle is a circle that passes through the centre of the given circle.

59. Hitherto the position of the fixed points and lines in the examples has generally been given with reference to the axes. The points and lines may however be given absolutely, with no statement as to their relation to the axes; or, to put it in another way, with no statement as to the axes we are to use. We are then left at liberty

to use such axes as we choose; generally the conditions of the problem will guide us in the choice. For simplicity, we use the given elements when possible.

Example i.—A point moves so that the product of its distances from two perpendicular lines is constant. Use the lines as axes.

Example ii.—A point moves so that the square of its distance from a fixed point varies as its distance from a fixed line through the point. Take the fixed point as origin, the line as one axis.

Frequently however it is not possible, or at any rate not advisable, to use the given elements in this way. For example, a point lies in the acute angle formed by two lines which make an angle 60°, and moves so that the product of its distances from the lines is constant. Since the lines are not at right angles, we cannot take them as the two axes unless we use a slightly different system of coordinates (oblique coordinates instead of rectangular). We might of course use *one* of the lines as an axis; but inasmuch as the lines are symmetrically involved in the statement of the problem it is desirable to use them symmetrically in its treatment. We select therefore as axes the two lines that bisect the angles formed by the given lines. If the axis of x is chosen to bisect the acute angle, then one of the given lines has inclination 30°, slope $\frac{1}{\sqrt{3}}$, and the other has inclination 150° (or $-30°$), slope $-\frac{1}{\sqrt{3}}$. Their equations are therefore—

$$y = \frac{1}{\sqrt{3}}.x,\ y = -\frac{1}{\sqrt{3}}.x,$$

that is, $$x - \sqrt{3}.y = 0,\ x + \sqrt{3}.y = 0.$$

Note.—In this manner we can always obtain the equation of two non-parallel lines in the form $mx - y = 0$, $mx + y = 0$.

Again, a point moves so that the sum of its distances from two fixed points is constant. If we choose one point for origin, we are treating the two points unsymmetrically, although they are symmetrically involved in the statement. It is better to take the origin half-way between the points; the line joining the points may then be taken for the axis of x. If the distance between the points is $2c$, their coordinates are now $(c, 0)$ and $(-c, 0)$.

The two objects to be attained are *simplicity* and *symmetry*. For the sake of simplicity we use given elements as origin and axes; but the more complicated the problem, the more important does symmetry become. Algebraic symmetry leads to algebraic simplicity.

60. In algebra, if we have two equations in x, y to be satisfied simultaneously (two simultaneous equations), we find a certain number of sets of solutions.

Example i.—
$$2x + 5y - 12 = 0,$$
$$7x - y - 5 = 0,$$
give one pair of values, $x = 1, y = 2$.

Example ii.—
$$7x - y - 5 = 0,$$
$$x^2 + y^2 + 6x - 4y - 3 = 0.$$

Here
$$y = 7x - 5,$$
hence
$$x^2 + (7x - 5)^2 + 6x - 4(7x - 5) - 3 = 0,$$
$$i.e.\ \ 25x^2 - 46x + 21 = 0.$$

Hence
$$x = 1,\ y = 7 - 5 = 2,$$
or
$$x = \frac{21}{25},\ y = \frac{147}{25} - 5 = \frac{22}{25};$$

there are two sets of solutions.

Example iii.— $x^2 + y^2 = 25,\ xy = 12.$

From these $(x + y)^2 = 49,\ (x - y)^2 = 1,$

$$\therefore\ x + y = \pm 7,\ x - y = \pm 1.$$

There are four sets of solutions—

$$x = 4,\ 3,\ -4,\ -3,$$
$$y = 3,\ 4,\ -3,\ -4.$$

The geometrical equivalent of a pair of values of x, y that satisfy two equations $f(x, y) = 0$, $F(x, y) = 0$ is a point common to the two curves represented by these equations. For any point whose coordinates satisfy $f(x, y) = 0$ is a point on the curve $f(x, y) = 0$, and any point whose coordinates satisfy $F(x, y) = 0$ is a point on the curve $F(x, y) = 0$. Thus to find the common points of two curves, we treat their equations as simultaneous and solve for x, y.

Example.— $x^2 + y^2 = 25,\ x - 4y + 13 = 0.$

$$(4y - 13)^2 + y^2 = 25,$$
$$17y^2 - 104y + 144 = 0,$$
$$(y - 4)(17y - 36) = 0.$$
$$y = 4 \text{ gives } x = 4y - 13 = 3,$$
$$y = \frac{36}{17} \text{ gives } x = \frac{144}{17} - 13 = \frac{144 - 221}{17} = -\frac{77}{17}.$$

The common points are $(3, 4)$, $\left(-\frac{77}{17}, \frac{36}{17}\right)$.

Notice that the value $y = 4$ substituted in $x^2 + y^2 = 25$ gives $x = \pm 3$, of which only $x = +3$ is available; and the value $y = \frac{36}{17}$ in $x^2 + y^2 = 25$ gives $x = \pm\frac{77}{17}$, of which only $x = -\frac{77}{17}$ satisfies the equation $x - 4y + 13 = 0$.

EXAMPLE.

Construct an accurate diagram to illustrate this example. Apply every step of the algebra to the diagram.

In solving two equations, of degree higher than the first, in two variables x, y, it is necessary to make sure that the values found do satisfy both equations; and the simplest available equation should be used for deducing the value of the second coordinate from the values found for the first.

Example.— $$x^2 + y^2 - 20x - 20y + 115 = 0,$$ $$x^2 + y^2 - 25 = 0.$$

By subtraction we obtain—

$$20x + 20y - 140 = 0,$$

that is, $$x + y - 7 = 0.$$

Hence $$x^2 + (x - 7)^2 - 25 = 0,$$

that is, $$2x^2 - 14x + 24 = 0,$$ $$x^2 - 7x + 12 = 0, \; x = 3 \text{ or } 4.$$

The value 3 substituted for x in $x^2 + y^2 - 25 = 0$ gives $y = +4$ or -4; while substituted in $x^2 + y^2 - 20x - 20y + 115 = 0$ it gives $y = +4$ or $+16$, and substituted in $x + y - 7 = 0$ it gives only $y = +4$.

Similarly the value 4 substituted for x in $x^2 + y^2 - 25 = 0$ gives $y = +3$ or -3; in $x^2 + y^2 - 20x - 20y + 115 = 0$, $y = +3$ or $+17$; in $x + y - 7 = 0$, $y = +3$.

The true solutions of the given equations are $x = 3$, $y = 4$, and $x = 4$, $y = 3$.

Thus the solutions found by the direct use of either of the given equations cannot be accepted without verification; but the use of the linear equation which presents itself in the course of the work avoids the introduction of irrelevant values. The reason for these other values is not difficult to see. The first of the given equations can be written—

$$(x - 10)^2 + (y - 10)^2 = 85,$$

which represents a circle, centre (10, 10), radius $\sqrt{85}$; the second represents a circle, centre O, radius 5. There are two points on the first circle for which $x = 3$, and two points on the second circle for which $x = 3$; but only one of these is common to the two circles. Similarly for $x = 4$.

We found however that the common points lie on the line $x + y - 7 = 0$ (see the first step in the solution of the equations). Now there is only one point on this line for which $x = 3$, namely, the point (3, 4), and only one point (4, 3) for which $x = 4$. Hence when we have found that the abscissæ of the common points are 3 and 4, we can find the ordinates from the equation $x + y - 7 = 0$ without the introduction of irrelevant values.

EXAMPLE.

Construct an accurate diagram to illustrate this example. Apply every step of the algebra to the diagram.

61. In this example from the equations of the two circles we obtain a simpler equation, which represents a straight line through the common points. In general, in solving two simultaneous equations we combine them algebraically, thus obtaining a new equation; this represents a new locus which passes through the points common to the two given loci. In the example above, one such equation was obtained by subtraction,

$$(x^2 + y^2 - 20x - 20y + 115) - (x^2 + y^2 - 25) = 0$$

gives

$$20(x + y - 7) = 0.$$

If the given equations are represented in the abridged

notation (§ 42) by $u = 0$, $v = 0$, then $u - v = 0$ is the equation of this straight line. Similarly the equations—

$$x^2 + y^2 - 25 = 0, xy - 12 = 0$$

give $$(x^2 + y^2 - 25) + 2(xy - 12) = 0,$$
that is, $$(x + y)^2 - 49 = 0,$$
or $$(x + y - 7)(x + y + 7) = 0,$$

which represents two straight lines through the common points.

Again $$(x^2 + y^2 - 25) - 2(xy - 12) = 0,$$
that is, $$(x - y)^2 - 1 = 0,$$

represents two other straight lines through the common points, viz.—

$$(x - y - 1)(x - y + 1) = 0.$$

Note.—These pairs of lines, in abridged notation, are—

$$u + 2v = 0, u - 2v = 0.$$

Example.

Construct an accurate diagram to illustrate this example. Apply every step of the algebra to the diagram.

62. In general, if $u = 0$, $v = 0$ are any two loci, then for all values of k (or of $\lambda : \mu$) the equation $u + kv = 0$ (or $\lambda u + \mu v = 0$) represents a locus through the common points (§ 45). For a point that lies on both loci has coordinates which make both expressions u and v assume the value 0, consequently $u + kv$ assumes the value 0, that is, the equation $u + kv = 0$ is satisfied. The locus thus found may of course break up into simpler loci, as in the example given above.

63. Precisely as the common points of two loci are found by treating their equations as simultaneous, so the common lines of two envelopes are found by treating the line-equations as simultaneous.

Example i.—Find the common tangents to the circles, centre O, radius 5; centre (12, 0), radius 1.

The tangents to the first circle satisfy—

$$\frac{1}{\sqrt{\xi^2 + \eta^2}} = 5\,; \quad \therefore \; \xi^2 + \eta^2 = \frac{1}{25};$$

the tangents to the second circle satisfy—

$$\frac{12\xi + 1}{\sqrt{\xi^2 + \eta^2}} = \pm 1\,; \quad \therefore \; \xi^2 + \eta^2 = (12\xi + 1)^2.$$

Hence to find the common tangents, solve the simultaneous equations—

$$\xi^2 + \eta^2 = \frac{1}{25}, \; \xi^2 + \eta^2 = (12\xi + 1)^2.$$

These give
$$(12\xi + 1)^2 = \frac{1}{25},$$

$$i.e. \quad 12\xi + 1 = \pm \frac{1}{5},$$

hence
$$\xi = -\frac{1}{10} \text{ or } -\frac{1}{15};$$

$$\eta^2 = \frac{1}{25} - \frac{1}{100} = \frac{3}{100}, \text{ or } \frac{1}{25} - \frac{1}{225} = \frac{8}{225},$$

$$\therefore \; \eta = \pm \frac{\sqrt{3}}{10} \text{ or } \pm \frac{2\sqrt{2}}{15}.$$

The equations are satisfied by all four sets—

$$\xi = \frac{1}{10}, \; -\frac{1}{10}, \; -\frac{1}{15}, \; -\frac{1}{15},$$

$$\eta = +\frac{\sqrt{3}}{10}, \; -\frac{\sqrt{3}}{10}, \; +\frac{2\sqrt{2}}{15}, \; -\frac{2\sqrt{2}}{15};$$

The equations of the common tangents are therefore—

$$-x + \sqrt{3}\,.\,y + 10 = 0,$$
$$-x - \sqrt{3}\,.\,y + 10 = 0,$$
$$-x + 2\sqrt{2}\,.\,y + 15 = 0,$$
$$-x - 2\sqrt{2}\,.\,y + 15 = 0.$$

EXAMPLE.

Show that $12\xi + 1 = +\frac{1}{5}$ gives exterior tangents;

$12\xi + 1 = -\frac{1}{5}$ gives interior tangents.

Example ii.—Find tangents from $(-1, 7)$ to the circle whose centre is O and radius 5. That is, find values of ξ, η to satisfy

$$-\xi + 7\eta + 1 = 0,$$

and
$$\frac{1}{\sqrt{\xi^2+\eta^2}} = 5, \text{ or } \xi^2 + \eta^2 = \frac{1}{25}.$$

These give
$$(7\eta + 1)^2 + \eta^2 = \frac{1}{25},$$
$$50\eta^2 + 14\eta + \frac{24}{25} = 0,$$
$$25^2\eta^2 + 7.25.\eta + 12 = 0,$$
$$(25\eta + 3)(25\eta + 4) = 0,$$
$$\therefore \eta = -\frac{3}{25} \text{ or } -\frac{4}{25};$$
$$\xi = 7\eta + 1 \text{ gives } \xi = \frac{4}{25} \text{ or } -\frac{3}{25}.$$

Hence the equations of the tangents are—

$$\frac{4}{25}x - \frac{3}{25}y + 1 = 0, \quad -\frac{3}{25}x - \frac{4}{25}y + 1 = 0,$$
$$i.e. \quad 4x - 3y + 25 = 0, \quad 3x + 4y - 25 = 0.$$

64. It is often necessary to find the lines that join some point, in particular the origin, to the points of intersection of a curve and a straight line. Now a line through O is $y - m_1x = 0$. Hence any number of lines, all passing through O, will be represented by an equation—

$$(y - m_1x)(y - m_2x)(y - m_3x) \ldots . (y - m_nx) = 0,$$

that is, by a homogeneous equation in x, y. Conversely, since by the theory of equations a homogeneous expression

in x, y of degree n can be written as the product of n factors of the type $y - mx$, it follows that a homogeneous equation of degree n in x, y represents n straight lines through the origin. In order then to find the lines that join the origin to the intersections of two loci, $u = 0$, $v = 0$, we combine the two equations in such a way as to produce a homogeneous equation, as shown in the following examples.

Example i.—Find the lines that join O to the common points of $fx^2 + gy^2 = 1$, $lx + my = 1$.

We are to combine the two equations so as to obtain a homogeneous equation. Now at the common points, $lx + my = 1$; hence if we write $lx + my$ for 1 wherever we choose in the equation $fx^2 + gy^2 = 1$, we shall obtain an equation that holds at these common points; for instance, $fx^2 + gy^2 = lx + my$. But since we wish to obtain a homogeneous equation, and since some of the terms in $fx^2 + gy^2 = 1$ are already of the second degree, we must make the remaining terms of the second degree also. Hence the equation wanted is $fx^2 + gy^2 = (lx + my)^2$.

Examine this equation. It is homogeneous of the second degree, and therefore represents two lines through O. Where $lx + my = 1$, we have also $fx^2 + gy^2 = 1$; that is, the two lines meet $lx + my = 1$ on the curve $fx^2 + gy^2 = 1$. These are therefore the lines that join the origin to the common points of $lx + my = 1$, $fx^2 + gy^2 = 1$.

Example ii.—Find the lines that join O to the common points of

$$lx + my + n = 0,\ ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

Here $-\dfrac{lx + my}{n} = 1$; we multiply the terms of the first degree, $2gx + 2fy$, by this expression of unit value, and the term c, of degree zero, by the square of this expression. We thus obtain the equation—

$$ax^2 + 2hxy + by^2 - \frac{lx + my}{n}(2gx + 2fy) + \left(\frac{lx + my}{n}\right)^2 c = 0,$$

i. e. $n^2(ax^2 + 2hxy + by^2) - 2n(lx + my)(gx + fy) + c(lx + my)^2 = 0$. This equation is homogeneous, of the second degree; it therefore represents two lines through O. Where these lines meet

$lx + my + n = 0$, that is, where $lx + my = -n$, the equation shows that—

$$n^2(ax^2 + 2hxy + by^2) + 2n^2(gx + fy) + cn^2 = 0,$$

that is,
$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

Hence the lines pass through the common points of this curve and $lx + my + n = 0$; they are therefore the required lines.

EXAMPLES.

1. Find the lines that join the origin to the common points of $x^2 + y^2 = 25$, $x + 2y + 5 = 0$.
2. Find the lines that join the origin to the common points of $y^2 = 4px$, $x + 3y + 2 = 0$.
3. Find the lines that join the origin to the common points of $x^2 + y^2 = r^2$, $px + qy + 1 = 0$.
4. The lines that join the centre of a circle of radius r to the common points of that circle and a straight line are at right angles. How far is the line from the centre of the circle?

65. It was found in algebra that some equations cannot be satisfied without the use of imaginary numbers; the roots of an equation (a quadratic, for example) may be real or they may be imaginary. Thus the roots of $x^2 - 4x + 3 = 0$ are real; the roots of $x^2 - 4x + 5 = 0$ are imaginary. The roots of $x^2 - px + q = 0$ are real or imaginary according as $p^2 - 4q$ is positive or negative; the roots of $ax^2 + 2bx + c = 0$ are real or imaginary according as $b^2 - ac$ is positive or negative. The question then arises, if the equations that present themselves in the analytical treatment of a geometrical problem have *imaginary* solutions, what is the meaning?

Let us take a problem in illustration.

Find a point whose distance from O shall be 5, and its distance from (26, 0) shall also be 5.

These conditions are expressed by the equations—

$$x^2 + y^2 = 25,$$
$$(x - 26)^2 + y^2 = 25,$$
$$\text{i.e.}\quad x^2 + y^2 - 52x + 26^2 = 25.$$

Hence $$52x - 26^2 = 0,$$
$$\therefore x = 13,$$

and $$y^2 = 25 - 13^2 = -144;\ \therefore y = \pm 12\sqrt{-1}.$$

Examine the diagram (Fig. 45). All points that satisfy the first condition lie on the circle (1); all that satisfy the second condition lie on the circle (2); the algebra shows that the points required lie also on the straight line $x = 13$. Obviously *no points of the diagram* satisfy these conditions.

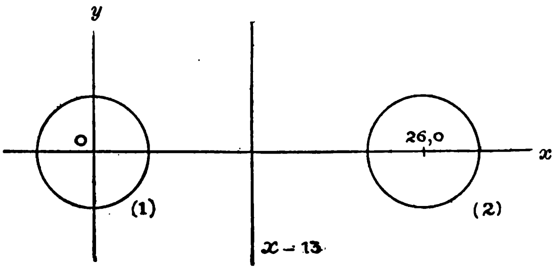


Fig. 45.

It would be quite correct to say that there are no points that satisfy the conditions. But then we shall have to say that sometimes the solutions of equations in x, y give coordinates of points, sometimes not. In numerical problems this would cause no difficulty; we could always tell, by the numerical values found, whether the algebraic

results had a geometrical application or not. But it would be inconvenient in problems that are not numerical. For instance, where does the line $y = mx + n$ meet the circle $x^2 + y^2 = r^2$? Algebra supplies the answer: the equations are satisfied by two sets of values of x, y; the values of x are the roots of the equation—

$$x^2(1 + m^2) + 2mnx + n^2 - r^2 = 0,$$

and the values of y are obtained from these by means of the relation $y = mx + n$. But in applying this to geometry we must add, there are two points, only if $(mn)^2 - (1 + m^2)(n^2 - r^2)$ is positive, none if this is negative. Thus we could never *assert* that pairs of values of x, y indicate points; it would be necessary to know what the values are. Literal algebra would be of little service; only numerical work would have any meaning.

It is found instead more convenient to make the geometry fit the algebra exactly, by speaking of *imaginary* points. We regard any pair of values of x, y as indicating a point; if the values of x, y are real, the point can be marked on the diagram; if the values of x or y, or of both x and y, are imaginary, the point cannot be marked on the diagram; we then speak of it as imaginary, using this name because the coordinates involve the imaginary quantity of algebra, $\sqrt{-1}$ or i.

66. The imaginary quantities involved enter only through the solution of algebraic equations in which it is necessary to take the square root of a negative quantity; hence if we find $f + if'$ as a solution, we shall find also

$f - if'$; for if' enters only as the square root of $-f'^2$, and this square root is $\pm\sqrt{-f'^2}$, that is, $\pm if'$. These two imaginary expressions, $f + if'$, $f - if'$, are called *conjugate*. Similarly the point whose coordinates are $f + if'$, $g + ig'$ has a conjugate, $f - if'$, $g - if'$.

The point half-way between two conjugate imaginary points is real; its coordinates are $\frac{1}{2}(f + if' + f - if')$, $\frac{1}{2}(g + ig' + g - ig')$, that is, f, g.

Notice that every imaginary point lies on one real line, which passes also through the conjugate imaginary point. For the line which joins $(f + if', g + ig')$ to $(f - if', g - ig')$ has the equation—

$$\frac{x - (f + if')}{f + if' - (f - if')} = \frac{y - (g + ig')}{(g + ig') - (g - ig')},$$

that is,
$$\frac{x - (f + if')}{2if'} = \frac{y - (g + ig')}{2ig'},$$

$$\frac{x - f - if'}{if'} = \frac{y - g - ig'}{ig'},$$

$$\frac{x - f}{if'} - 1 = \frac{y - g}{ig'} - 1,$$

$$\frac{x - f}{if'} = \frac{y - g}{ig'},$$

that is,
$$\frac{x - f}{f'} = \frac{y - g}{g'},$$
which is real. It is the line through (f, g) with the slope $\frac{g'}{f'}$.

It is clear that there cannot be *two* real lines through an imaginary point, for two real lines have a real intersection, either at a finite distance or at infinity.

67. Similarly it is found advisable to consider imaginary lines, lines whose coordinates are imaginary. If the coordinates of a line are imaginary, then the coefficients in the equation of the line are imaginary; for instance, the line whose coordinates are $2 + i, 3 - 4i$ has the equation—

$$(2 + i)x + (3 - 4i)y + 1 = 0,$$

which can be written with a real coefficient of x (multiply by $2 - i$) or y (multiply by $3 + 4i$)—

$$5x + (2 - 11i)y + 2 - i = 0,$$
$$(2 + 11i)x + 25y + 3 + 4i = 0.$$

The general equation of an imaginary line can be written—

$$(a + ia')x + (b + ib')y + (c + ic') = 0,$$

where a, b, c, a', b', c' are real quantities. The conjugate is then—

$$(a - ia')x + (b - ib')y + (c - ic') = 0.$$

The point of intersection of these two lines is real; for in solving these equations, we combine them by addition and subtraction, obtaining—

$$2ax + 2by + 2c = 0,$$
$$2ia'x + 2ib'y + 2ic' = 0;$$

that is, $ax + by + c = 0$, $a'x + b'y + c' = 0$, which give for x, y the real values—

$$\frac{bc' - b'c}{ab' - a'b}, \quad \frac{ca' - c'a}{ab' - a'b}.$$

[Or else, as in dealing with imaginary points, the line

$(f + if', g + ig')$ and the line $(f - if', g - ig')$ determine a real point whose equation is $\dfrac{\xi - f}{f'} = \dfrac{\eta - g}{g'}$.]

EXAMPLES.

1. Find the real line through (i) $(5 + 3i, 5 + 4i)$, (ii) $(2 + 8i, 1 - 6i)$. Show that the two lines are at right angles, and find their common point.

2. Show that the two points $(5 + 3i, 5 + 4i)$, $(2 + 3i, 1 + 4i)$ lie on the same real line. How far apart are the points?

3. Show that every point on either of the lines $x + iy = 0$, $x - iy = 0$ is at zero distance from the origin.

CHAPTER V

CONICS

68. A CONIC is the locus of a point which moves so that its distance from a fixed point is in a constant ratio to its distance from a fixed line. The fixed point is called the focus, the fixed line is called the directrix; the constant ratio is called the eccentricity, and is denoted by e.

If the fixed point be (p, q), and the fixed line be $lx + my + n = 0$, the given condition is expressed algebraically by—

$$\sqrt{(x-p)^2 + (y-q)^2} = e \cdot \frac{lx + my + n}{\pm\sqrt{l^2 + m^2}},$$

that is, by—

$$(x-p)^2 + (y-q)^2 = \frac{e^2}{l^2 + m^2}(lx + my + n)^2.$$

Thus the equation of the locus is of the second degree.

EXAMPLES.

1. Find the equation of the conic with focus $(1, 0)$, directrix $x - 4 = 0$, $e = \frac{1}{2}$.

2. Find the equation of the conic with focus $(-1, 0)$, directrix $x + 4 = 0$, $e = \frac{1}{2}$.

3. Find the equation of the conic with focus (3, 3), directrix $x + y - 3 = 0$, $e = \sqrt{2}$.

4. Find the equation of the conic with focus (5, 0), directrix $x + 5 = 0$, $e = 1$.

69. It is clear that a proper choice of axes may possibly simplify the form of the equation. It will now be shown that the equation can thus be obtained in one of the following forms—

$$y^2 = 4px, \text{ if } e = 1,$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ if } e < 1,$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \text{ if } e > 1.$$

Later on (Chap. XI.) it will be shown that every equation of the second degree, that does not break up into two of the first degree, can be reduced by change of axes to one of these forms. That is, it will be shown that an equation of the second degree represents either a *pair of straight lines* or a *conic.*

Thus conics come immediately after straight lines in the algebraic treatment; for all equations of the first degree (in point-coordinates) represent straight lines, and the next equations to take up are naturally those of thesecond degree.

Note.—The general equation of the second degree will be written—

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

70. If $e = 1$, the conic is called a parabola. A parabola can therefore be defined either as a conic whose eccentricity is unity, or as the locus of a point which moves so that

its distance from a fixed point is equal to its distance from a fixed line.

Take as axis of x the line through the focus perpendicular to the directrix, and as origin the point on this line half-way between the focus and directrix. Then if the distance from the directrix to the focus is denoted by $2p$, the coordinates of the focus are p, 0, and the equation of the directrix is $x + p = 0$. The equation of the parabola is now—

$$\sqrt{(x-p)^2 + y^2} = x + p,$$

that is,

$$y^2 = 4px.$$

The double ordinate through the focus is called the latus rectum. Since the focus is $(p, 0)$, the semi-latus rectum is the value of y given by the equation $y^2 = 4px$ when $x = p$, that is, $2p$. Hence the latus rectum is equal to $4p$, and lies on the line whose equation is $x = p$.

71. The conic is called an ellipse if $e < 1$, a hyperbola if $e > 1$. The line through the focus perpendicular to the directrix is evidently an axis of symmetry; take this line as axis of x. Although the statement of the problem suggests the directrix as suitable for the axis of y, this does not lead to the simplest form for the equation; instead, we choose as origin a certain point C, whose relation to the given elements (focus and directrix) will appear shortly. The coordinates of the focus are now c, 0, and the equation of the directrix is $x = d$, where the quantities c, d are still to be determined; they are connected by one relation, $c - d = f$, where f is the distance from the directrix to

the focus. In Fig. 46, S is the focus, XM is the directrix, met by the axis of symmetry at the point X, usually called the foot of the directrix; CX $= d$, CS $= c$, XS $= f$.

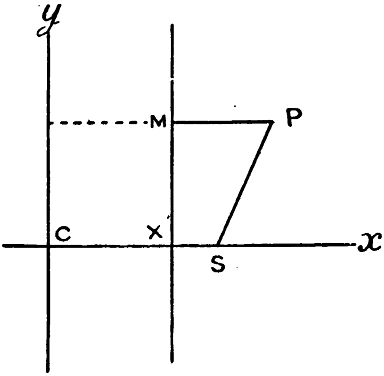


Fig. 46.

By the definition of the conic—

$$SP = e \, . \, MP,$$

hence $\sqrt{(x-c)^2 + y^2} = e(x-d),$

therefore $x^2 - 2cx + c^2 + y^2 = e^2x^2 - 2e^2dx + e^2d^2.$

If $e < 1$, this is written—

$$(1-e^2)x^2 + y^2 + 2x(e^2d - c) + (c^2 - e^2d^2) = 0,$$

and if $e > 1$,

$$(e^2-1)x^2 - y^2 - 2x(e^2d - c) - (c^2 - e^2d^2) = 0.$$

To simplify this equation in either of its forms, choose the origin so that $e^2d - c = 0$; this gets rid of one term. Geometrically this determines the position of C by means

of the ratio in which C divides the line SX; for CS : CX $= c : d = e^2 : 1$.[1] Algebraically the position of C is determined by means of the values of c and d, obtained from the equations—

$$c - d = f,$$
$$c - e^2 d = 0.$$

These give
$$c(1 - e^2) = -e^2 f,$$
$$d(1 - e^2) = -f,$$

which, if $e > 1$, are better written—

$$c(e^2 - 1) = e^2 f,$$
$$d(e^2 - 1) = f.$$

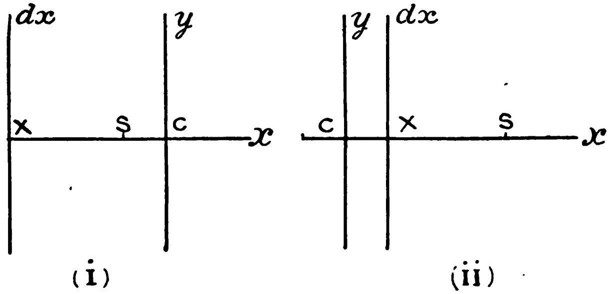


Fig. 47.

The relative position of the points is shown[2] in Fig. 47; in 47 (i), $e < 1$; in 47 (ii), $e > 1$.

[1] This can be stated also in the form (§ 7), SC : CX $= -e^2 : 1$.

[2] The values of e in Fig. 47 are $\frac{1}{2}$ and 2; other values will change the distances of the points, but not their order on the axis of symmetry.

With these values for c, d the absolute term in the equation, $c^2 - e^2d^2$, becomes—

$$\begin{aligned}
&\frac{1}{(1-e^2)^2}(e^4f^2 - e^2f^2),\\
&= \frac{f^2(e^4 - e^2)}{(1-e^2)^2},\\
&= -\frac{e^2f^2(1-e^2)}{(1-e^2)^2},\\
&= -\frac{e^2f^2}{1-e^2}.
\end{aligned}$$

The equation of the ellipse is therefore—

$$(1-e^2)x^2 + y^2 = \frac{e^2f^2}{1-e^2},$$

that is,
$$\frac{x^2}{\dfrac{e^2f^2}{(1-e^2)^2}} + \frac{y^2}{\dfrac{e^2f^2}{1-e^2}} = 1;$$

and the equation of the hyperbola is—

$$(e^2-1)x^2 - y^2 = \frac{e^2f^2}{e^2-1},$$

or
$$\frac{x^2}{\dfrac{e^2f^2}{(e^2-1)^2}} - \frac{y^2}{\dfrac{e^2f^2}{e^2-1}} = 1.$$

Instead of retaining the given quantities e, f, we substitute other quantities a, b, for the two combinations of e, f that are present in these equations. For the ellipse, we write a^2, b^2 for the two positive quantities—

$$\frac{e^2f^2}{(1-e^2)^2}, \quad \frac{e^2f^2}{1-e^2};$$

and for the hyperbola we write a^2, b^2 for the two positive quantities—

$$\frac{e^2f^2}{(e^2-1)^2}, \frac{e^2f^2}{e^2-1}.$$

The equations are now—

ellipse— $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

hyperbola— $$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

Note.—The expressions $\frac{e^2f^2}{(1-e^2)^2}$, etc., are numerical multiples of f^2, where f represents a line; these expressions are therefore of *two dimensions in lines.* Hence a^2, b^2 are of two dimensions in lines, and a, b themselves are lines.

72. From the form of the equations thus obtained, we learn that the axis of y is an axis of symmetry, as well as the axis of x. Hence a conic of any given eccentricity, different from unity, described with a focus and directrix on the right of the axis of y (the second axis of symmetry) can equally well be described with a focus and directrix similarly placed on the left of this axis. If one focus is $(c, 0)$, and the directrix $x-d=0$, the other focus is $(-c, 0)$, and the directrix is $x+d=0$.

Moreover, the point C is a centre of symmetry, for it is the intersection of two perpendicular axes of symmetry. It is called the centre of the ellipse, or hyperbola. The ellipse and hyperbola are the *central* conics.

The axis of symmetry on which the two foci lie is called the transverse axis; the other axis of symmetry is the conjugate axis.

The points of intersection of a conic with an axis of symmetry are called vertices of the curve.

The ellipse and hyperbola meet the transverse axis ($y = 0$) where $x^2 = a^2$, $x = \pm a$. The length $2a$, the part of the transverse axis intercepted by the curve, is called the major axis.

The ellipse meets the conjugate axis ($x = 0$) where $y^2 = b^2, y = \pm b$. The length $2b$, the part of the conjugate axis intercepted by the ellipse, is called the minor axis. The hyperbola meets its conjugate axis where $y^2 = -b^2$, hence at imaginary points; the hyperbola does not intercept any real length on the conjugate axis, but by analogy with the case of the ellipse, the length $2b$ is called the minor axis of the hyperbola. (Compare § 132.)

Note.—The transverse axis and the conjugate axis are the *lines* of indefinite extent; the major axis and the minor axis are properly definite *segments* of these lines, but occasionally the terms major axis and minor axis are used in both senses.

73. In the geometry of the ellipse and hyperbola, the important lengths are a, b, c; the lengths d and f are at once expressible in terms of these. It is necessary to have the relations that connect these lengths clearly in mind.

(i) The ellipse (Fig. 48). $e < 1$.

Fig. 48.

Inasmuch as there is a focus on each side of C, there is one focus for which the abscissa is positive; it is better to

take this one as S, instead of the left-hand one as in Fig. 47 (i); hence instead of the negative value found for c, we now have a positive value, namely, $c = \frac{e^2 f}{1-e^2}$. We have also, by definitions of a^2 and b^2,

$$a^2 = \frac{e^2 f^2}{(1-e^2)^2},\ b^2 = \frac{e^2 f^2}{1-e^2}.$$

Hence $$a^2 - b^2 = c^2,$$

and $$c = ae;$$

also $$e^2 = \frac{a^2 - b^2}{a^2}.$$

Moreover, since $$\mathrm{CX}' = -\frac{f}{1-e^2},$$

$$\therefore\ \mathrm{CX} = +\frac{f}{1-e^2} = \frac{a}{e},$$

and the equation of the directrix is therefore $x = \frac{a}{e}$, that is, $ex - a = 0$. This last is easily remembered by means of the relation $\mathrm{CS}\,.\,\mathrm{CX} = \mathrm{CA}^2$.

Hence when the semi-axes a and b are given, the position of the foci and directrices, and the value of the eccentricity, can be found at once. Notice that the foci lie on the longer axis; for since $a^2 - b^2$ is positive, $= c^2$, it is seen that a is greater than b in the case of the ellipse.

(ii) The hyperbola (Fig. 49). $e > 1$.

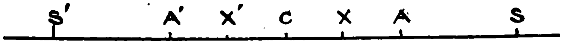


Fig. 49.

In this case $a^2 = \frac{e^2 f^2}{(e^2-1)^2},\ b^2 = \frac{e^2 f^2}{e^2 - 1},\ c = \frac{ef}{e^2-1}.$

Hence $$a^2 + b^2 = c^2,$$
and $$c = ae;$$
also $$e^2 = \frac{a^2 + b^2}{a^2}.$$

Moreover $\mathrm{CX} = \dfrac{f}{e^2 - 1} = \dfrac{a}{e}$, and the equation of the directrix is $ex - a = 0$. Hence, as in the case of the ellipse, $\mathrm{CS} \, . \, \mathrm{CX} = \mathrm{CA}^2$.

Comparison of these formulæ for the ellipse and hyperbola shows that the only difference is in the sign of b^2. If we write α for a^2, β for $+ b^2$ in the ellipse, and for $- b^2$ in the hyperbola, so that the equations of both curves become—

$$\frac{x^2}{\alpha} + \frac{y^2}{\beta} = 1,$$

the formulæ of this section become—

$$c^2 = \alpha - \beta, \; e^2 = \frac{\alpha - \beta}{\alpha}. \quad (\text{Cf. § 79.})$$

The double ordinate through the focus is called the latus rectum; it is twice the value of y given by the equation when for x is written c. For the ellipse,

$$\frac{y^2}{b^2} = 1 - \frac{c^2}{a^2}$$
$$= 1 - \frac{a^2 - b^2}{a^2} = \frac{b^2}{a^2}.$$

Hence the latus rectum $= 2\,\dfrac{b^2}{a}$.

For the hyperbola,

$$\frac{y^2}{b^2} = \frac{c^2}{a^2} - 1$$

$$= \frac{a^2 + b^2}{a^2} - 1 = \frac{b^2}{a}.$$

Hence the latus rectum of the hyperbola, as of the ellipse, $= 2\frac{b^2}{a}$.

In the case of the hyperbola the difference between the axes is not that one is greater than the other, as for the ellipse, but that one meets the curve in real points, the other in imaginary points. The foci lie on the axis that meets the curve in *real* points.

The terms major and minor, as here used, do not mean greater and less. It is true that the major axis of an ellipse is necessarily the greater axis; but the major axis of a hyperbola may be less than the minor axis.

There is no special reason why the major axis should lie along the axis of x; it may equally well lie along the axis of y. Similarly the axis of a parabola may equally well lie along Oy; the equation can then be obtained in the form $x^2 = 4py$.

74. If the axes of an ellipse are equal ($a = b$), the equation reduces to $x^2 + y^2 = a^2$, the equation of a circle; hence the circle can be looked upon as a particular case of the ellipse. Since $c^2 = a^2 - b^2 = 0$, the two foci come together; also $e = \frac{c}{a} = 0$, therefore the eccentricity is zero, and the directrix, whose equation is $x = \frac{a}{e}$, is at infinity. Thus

the circle cannot be obtained, practically, by the focus and directrix property; nevertheless its equation is included under that of the ellipse, and is obtained by writing $b = a$.

When the centre of the circle is not at the origin, but at a point (p, q), the equation of the circle of radius r is—

$$(x - p)^2 + (y - q)^2 = r^2,$$

that is, $$x^2 + y^2 - 2px - 2qy + p^2 + q^2 - r^2 = 0.$$

This is an equation of the second degree, but of a special form; it contains no term xy, and the coefficients of the terms x^2, y^2 are equal, whereas the general equation of the second degree is—

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

If $h = 0$ and $a = b$, the equation represents a circle. To prove this, divide by a, the equation is then—

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$

where g, f, c are written for the original coefficients divided by a. By completing the squares in x and y, we can write this—

$$(x + g)^2 + (y + f)^2 = g^2 + f^2 - c.$$

If C be the point $(-g, -f)$, and P the point (x, y), this equation states that—

$$\mathrm{CP}^2 = g^2 + f^2 - c,$$

that is, $$\mathrm{CP} = \sqrt{g^2 + f^2 - c}.$$

Hence the locus of P is a circle, with centre $\mathrm{C}(-g, -f)$, and radius $= \sqrt{g^2 + f^2 - c}$.

This equation, $x^2 + y^2 + 2gx + 2fy + c = 0$, is the *general equation of a circle.*

EXAMPLES.

1. Find the major axis and minor axis of the following conics; state which axis lies along the axis of x; find the coordinates of the real vertices and the foci, and the equations of the directrices; and find the value of the eccentricity. Give the name of each curve.

$$\text{(i)}\ 4x^2 + 9y^2 = 36.$$
$$\text{(ii)}\ \frac{x^2}{4} + \frac{y^2}{9} = 1.$$
$$\text{(iii)}\ \frac{x^2}{4} + \frac{y^2}{9} = 25.$$
$$\text{(iv)}\ x^2 - 4y^2 = 1.$$
$$\text{(v)}\ x^2 - 4y^2 - 4 = 0.$$
$$\text{(vi)}\ x^2 - 4y^2 + 4 = 0.$$
$$\text{(vii)}\ x^2 + 8y = 0.$$
$$\text{(viii)}\ 9x^2 + 4y = 0.$$

2. Verify your results for these eight curves by forming the equation of a conic of this eccentricity, with one of the foci and the corresponding directrix.

3. Prove that the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is the locus of a point which moves so that the sum of its distances from the two foci is equal to the major axis.

Explain the particular case that arises when the two foci come together.

4. Prove that the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is the locus of a point which moves so that the difference of its distances from the two foci is equal to the major axis.

5. Prove that in Exs. 1–7, and in Ex. 11, on pp. 85, 86, the point P describes a conic. State in each case the nature of the conic, and the position of a focus and directrix, or of the two foci.

75. The general shape of the different conics can be seen from the focus and directrix definitions, or, as regards the ellipse (and hyperbola), from the property that the sum (or difference) of the focal distances is constant, a

property which can be utilised for drawing the curve by simple mechanical means. (See Exs. 1 and 11, p. 85, 86.) The curves can also be drawn as accurately as desired by plotting the equations.

(i) Parabola. The equation is $y^2 = 4px$. Take for y a series of values, and then obtain the values of x, by means of the relation $x = \frac{y^2}{4p}$. Notice that the same value of x is obtained from two opposite values of y,

$$\begin{array}{llllll} \text{e.g.}\quad y = 0, & \pm p, & \pm 2p, & \pm 3p, & \pm 4p, & \text{etc.} \\ \quad\quad\;\; x = 0, & \frac{p}{4}, & p, & \frac{9p}{4}, & 4p, & \text{etc.} \end{array}$$

There are no restrictions on the value of y; it may

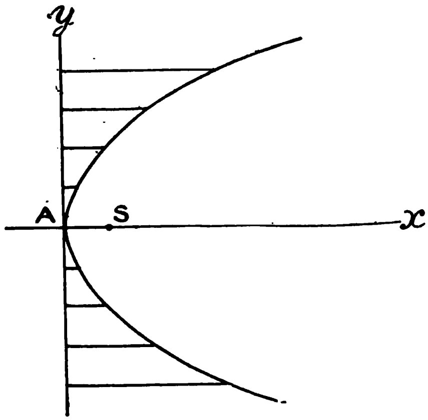


Fig. 50.

increase indefinitely. The curve passes off to infinity (Fig. 50).

Notice that the parabola has only one vertex, which lies at the origin with the equation we are using; and that the focus is $(p, 0)$. The ordinate of the parabola at the focus is $2p$, that is, the ordinate $= 2 \times$ abscissa.

(ii) Ellipse. From the equation—

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

we have
$$y = \pm \frac{b}{a}\sqrt{a^2 - x^2}.$$

This shows that values of $x^2 > a^2$ make y imaginary; while values of $x^2 \leq a^2$ give two equal and opposite real values for y. Similarly the equation $x = \pm \frac{a}{b}\sqrt{b^2 - y^2}$ shows that y^2 cannot exceed the value b^2, and attains this value only when $x = 0$. The curve lies therefore entirely within a rectangle of length $2a$, breadth $2b$.

For calculating numerical values, it is better to write the expression for y in the form $\frac{y}{b} = \pm\sqrt{1 - \left(\frac{x}{a}\right)^2}$; then give to $\frac{x}{a}$ a series of values numerically less than unity, each of which yields a pair of values $(\pm)$ for y; in this way as many points as desired can be found (Fig. 51); and thus the form of the curve is discovered—

$$\text{e.g.}\quad \frac{x}{a} = \pm 1,\ \pm\frac{12}{13},\ \pm\frac{4}{5},\ \pm\frac{3}{5},\ \pm\frac{5}{13},\quad 0;$$

$$\frac{y}{b} = \quad 0,\ \pm\frac{5}{13},\ \pm\frac{3}{5},\ \pm\frac{4}{5},\ \pm\frac{12}{13},\ \pm 1.$$

If S is a focus, CS $= c$, where $c^2 = a^2 - b^2$ (§ 73).

Hence
$$\begin{aligned} BS^2 &= BC^2 + CS^2 \\ &= b^2 + c^2 \\ &= a^2; \end{aligned}$$

therefore
$$BS = a = CA = BR.$$

This shows that a circle described with centre B, to pass

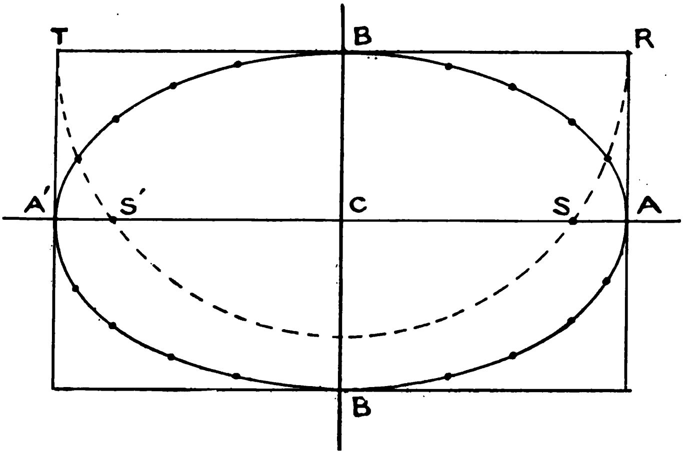


Fig. 51.

through the nearest vertices (R, T) of the rectangle determined by the axes of the curve, passes through the two foci.

(iii) Hyperbola. Since $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$,

$$\therefore \frac{x}{a} = \pm\sqrt{1 + \left(\frac{y}{b}\right)^2},$$

and
$$\frac{y}{b} = \pm\sqrt{\left(\frac{x}{a}\right)^2 - 1}.$$

The expression for x shows that y may assume any value, positive or negative, from zero to infinity; x is always real. From the expression for y it is seen that $\left(\frac{x}{a}\right)^2$ must be ≥ 1; if $\left(\frac{x}{a}\right)^2 < 1$, y is imaginary. Hence the curve is excluded from the part of the plane that lies between the lines $x = a$, $x = -a$; it stretches from this excluded region to infinity in both directions.

Construct the rectangle whose sides are $2a$ and $2b$, and also its diagonals, the lines $\frac{x}{a} - \frac{y}{b} = 0$, $\frac{x}{a} + \frac{y}{b} = 0$. These diagonals are a help in drawing the curve, for, as will be

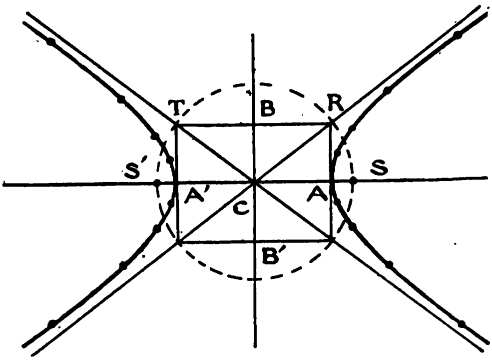


Fig. 52.

shown in Chapter IX., the curve continually approaches these lines, and is ultimately (at infinity) indefinitely close to them.

From the equation $\frac{y}{b} = \pm\sqrt{\left(\frac{x}{a}\right)^2 - 1}$ we obtain by

calculation as many pairs of values as desired, and thus the curve is traced,

$$\text{e.g.}\quad \frac{x}{a} = \pm 1, \pm \frac{13}{12}, \pm \frac{5}{4}, \pm \frac{5}{3}, \pm \frac{13}{5}, \text{ etc.}$$

$$\frac{y}{b} = \quad 0, \pm \frac{5}{12}, \pm \frac{3}{4}, \pm \frac{4}{3}, \pm \frac{12}{5}, \text{ etc.}$$

If S is a focus, CS $= c$, where $c^2 = a^2 + b^2$ (§ 73.)

Hence
$$\begin{aligned} CS^2 &= CA^2 + AR^2 \\ &= CR^2. \end{aligned}$$

This shows that a circle described with centre C, to pass through the vertices of the rectangle, passes through the two foci.

Example.

Trace carefully the conics whose equations are given in Ex. 1 p. 120. Mark the foci and directrices.

CHAPTER VI

RELATION OF STRAIGHT LINES TO CURVES

76. THE problems of Chapters IV. and V. led to equations of various degrees: the circle, parabola, ellipse, and hyperbola have equations of the second degree in point-coordinates; other loci have equations of degree 3, 4, etc. All the loci whose equations are of the second degree have this fact in common—they are met by a straight line in precisely two points. To prove this, we combine with the equation of the curve the equation of a straight line $y = mx + n$, and thus obtain an equation for the coordinates of the points of intersection.

Example.—The parabola $y^2 = 4px$ meets the line $y = mx + n$, where
$$(mx + n)^2 = 4px,$$
that is, where $$m^2x^2 + 2(mn - 2p)x + n^2 = 0;$$
there are therefore two common points, whose abscissæ are the roots of this quadratic equation. Similarly the line $y = mx + n$ meets the circle $x^2 + y^2 = r^2$, where $x^2 + (mx + n)^2 = r^2$; it meets the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, or the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $\frac{x^2}{a^2} \pm \frac{(mx+n)^2}{b^2} = 1$.

The algebraic fact is that the equation of the curve is of the second degree; to express the geometrical equivalent, that two points belonging to the locus lie on an arbitrary line, we say that the locus is of the *second order*.

To express the corresponding fact as regards an envelope whose line-equation is of the second degree, namely that two lines of the system pass through an arbitrary point, we say that the envelope is of the *second class*.

An equation of the third degree in point-coordinates represents a curve of the third order. For example, $y = x^3$ meets $y = mx + n$ where $x^3 - (mx + n) = 0$. By the theory of equations this has three roots; hence the line and the curve have three common points. Similarly, in general, an equation of degree p in x, y combined with $y = mx + n$ gives rise to an equation of degree p in x; this, by the theory of equations, has p roots, hence the locus (or curve) is of order p.

EXAMPLES.

1. Find the common points of $4x + y - 12 = 0$ and $y^2 = 8x$.
2. Find the common points of $2x + y - 7 = 0$ and $x^2 + y^2 = 13$.
3. Find where $y = 4x$ meets $y = x^3$.
4. Find the third point at which the line through $(2, 8)$, $(-1, -1)$ meets $y = x^3$.
5. Represent (3) and (4) on an accurate diagram.

77. The two points at which a line meets a curve of the second order, for instance a circle, are determined by means of a quadratic equation; hence—

(i) The points may be real and distinct, the line then visibly cuts the curve.

(ii) The two values found for x may be equal.

(iii) The points may be imaginary, then the line does not appear to meet the curve.

For example, $y = \frac{3}{4}x + n$

meets $x^2 + y^2 = 25$

where $x^2 + \frac{9}{16}x^2 + \frac{3}{2}nx + n^2 = 25,$

i. e. where $\frac{25}{16}x^2 + \frac{3}{2}nx + n^2 - 25 = 0.$

The roots of this equation in x are real and different if

$$\left(\frac{3n}{2}\right)^2 < 4 \cdot \frac{25}{16}(n^2 - 25),$$

that is, if $9n^2 > 25n^2 - 625$, therefore if $16n^2 < 625$; they are imaginary if $16n^2 > 625$.

If $16n^2 = 625$, that is, if $n = \pm \frac{25}{4}$, the roots of the equation are equal; the line $y = \frac{3}{4}x + \frac{25}{4}$ meets the circle where

$$\frac{25}{16}x^2 + \frac{75}{8}x + \frac{225}{16} = 0,$$

that is, where $x^2 + 6x + 9 = 0,$

or $(x + 3)^2 = 0.$

Thus we obtain only the value -3 for x, with the corresponding value $+4$ for y; that is, the line $y = \frac{3}{4}x + \frac{25}{4}$ meets the circle only at $(-3, 4)$, and similarly the line $y = \frac{3}{4}x - \frac{25}{4}$ meets the circle only at $(3, -4)$.

In algebra we do not say that the quadratic equation has only one root; we say that the two roots are equal. Similarly in analytical geometry we do not say that a line meets a circle sometimes in two points, sometimes in one, we say instead that a line always meets a circle in two points, but these may be indistinguishable (or coincident). It is clear in this example that the line $y = \frac{3}{4}x + \frac{25}{4}$ is a

tangent, as defined in elementary geometry; it meets the circle at $(-3, 4)$ and nowhere else. A similar relation is possible for any curve of the second order; for example, the line $y = \frac{1}{2}x + 6$ meets the parabola $y^2 = 12x$ where $\left(\frac{1}{2}x + 6\right)^2 = 12x$, that is, where $x^2 + 24x + 144 = 48x$, or $(x - 12)^2 = 0$, from which $x = 12$. Since $y = \frac{1}{2}x + 6$, the corresponding value of y is 12; the line $y = \frac{1}{2}x + 6$ meets the parabola only at the point $(12, 12)$. Similarly the equation that gives the abscissæ of the common points of $y = x + \sqrt{41}$ and $\frac{x^2}{25} + \frac{y^2}{16} = 1$ is $(\sqrt{41} \,.\, x + 25)^2 = 0$; hence the line meets the ellipse only at $\left(-\frac{25}{\sqrt{41}}, \frac{16}{\sqrt{41}}\right)$.

EXAMPLES.

1. Show that the lines $x - y + 3$, $3x - y + 1 = 0$, meet $y^2 = 12x$ at coincident points.
2. Show that the line $3x - 4y + 20 = 0$ meets each of the curves $y^2 = 15x$, $x^2 + y^2 = 16$ at coincident points.
3. Show that the circles $x^2 + y^2 = 25$, $x^2 + y^2 - 4x - 3y = 0$ meet at coincident points.

78. In each of these cases the line meets the curve only at the one point, and is therefore a tangent. But consider another example. The line $y = 3x - 2$ meets the curve $y = x^3$ where $x^3 - 3x + 2 = 0$, that is, where $(x - 1)^2(x + 2) = 0$. From $x = 1$ we find $y = 1$, and from $x = -2$, $y = -8$. The three points at which the line

meets the curve in one at $(-2, -8)$, *two* at $(1, 1)$. The line is clearly a tangent to the curve at P (Fig. 53), yet it does meet the curve elsewhere. Thus the definition of a tangent used in elementary geometry is not satisfactory for curves of order higher than the second. We use instead a definition which expresses that two points of intersection become indistinguishable, but makes no statement about the remaining intersections.

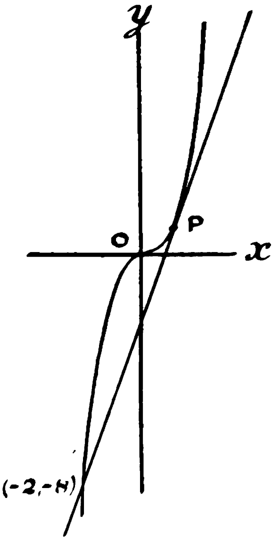


Fig. 53.

Definition of a tangent.—Let P be a point on a curve, Q a neighbouring point, PQ the secant; let Q move along the curve towards P. The line PQ, in the limiting position which it assumes[1] as Q becomes indistinguishable from P, is the tangent at P.

Definition of a normal.—The line through P perpendicular to the tangent at P is called the normal at P.

The condition that a line $y = mx + n$ be a tangent to a curve of the second order is found by expressing that the two roots of the quadratic are equal. If the curve is of the third order, the equation for intersections is of the third

[1] If there *is* a limiting position, which is not always the case. That PQ has a limiting position for the curves here considered is shown in the process for finding the tangent at a point (Chap. VII).

degree; we require therefore the condition that two roots of a cubic equation be equal, etc. For the quadratic $ax^2 + 2bx + c = 0$, the condition is $b^2 = ac$ (which for $x^2 + px + q = 0$ takes the form $p^2 = 4q$). For the cubic $ax^3 + 3bx^2 + 3cx + d = 0$, it is shown in works on the theory of equations that the condition is $(ad - bc)^2 = 4(ac - b^2)(bd - c^2)$.

79. To find the condition that a line $y = mx + n$ be a tangent to a given conic, form the equation for intersections, which will be a quadratic; then write down the condition that the two roots of this quadratic be equal. This will give a relation connecting m and n.

(i) *Parabola.*—The line $y = mx + n$
meets the parabola $y^2 = 4px$
where $(mx + n)^2 = 4px$;
the equation for intersections is therefore—

$$m^2x^2 + 2(mn - 2p)x + n^2 = 0.$$

The roots of this are equal if

$$(mn - 2p)^2 = m^2n^2,$$

that is, if $m^2n^2 - 4mnp + 4p^2 = m^2n^2$,

which reduces to $mn = p$.

This relation expresses either of the two quantities m, n in terms of the other, e.g. $n = \frac{p}{m}$. The quadratic then becomes—

$$m^2x^2 - 2px + \frac{p^2}{m^2} = 0,$$

that is, $$\left(mx - \frac{p}{m}\right)^2 = 0.$$

Hence $x = \frac{p}{m^2}$, $y = mx + n = m \cdot \frac{p}{m^2} + \frac{p}{m} = \frac{2p}{m}$. Thus for all values of m, the line $y = mx + \frac{p}{m}$ is a tangent to $y^2 = 4px$, and the point of contact is $\left(\frac{p}{m^2}, \frac{2p}{m}\right)$. Notice that for a given slope, m, there is only one tangent to a parabola.

(ii) *Circle.*—Similarly we can find the relation between m and n that makes $y = mx + n$ a tangent to the circle $x^2 + y^2 = r^2$. The equation for intersections is—

$$(m^2 + 1)x^2 + 2mnx + n^2 - r^2 = 0;$$

the roots of this are equal if

$$(mn)^2 = (m^2 + 1)(n^2 - r^2),$$

that is, if

$$n^2 = r^2(m^2 + 1),$$

$$n = \pm r\sqrt{m^2 + 1}.$$

The point of contact is obtained by solving the quadratic, using this value of n. If $n = + r\sqrt{m^2 + 1}$, the equation becomes—

$$(m^2 + 1)x^2 + 2mr\sqrt{m^2 + 1}\, x + m^2r^2 = 0,$$

that is,

$$(\sqrt{m^2 + 1} \, . x + mr)^2 = 0.$$

Hence

$$x = -\frac{mr}{\sqrt{m^2 + 1}}, \; y = \frac{r}{\sqrt{m^2 + 1}}.$$

Thus for a given slope m there are two tangents; one is $y = mx + r\sqrt{m^2 + 1}$, and the point of contact is

$\left(\frac{-mr}{\sqrt{m^2+1}}, \frac{r}{\sqrt{m^2+1}}\right)$; the other is $y = mx - r\sqrt{m^2+1}$, and the point of contact is $\left(\frac{+mr}{\sqrt{m^2+1}}, \frac{-r}{\sqrt{m^2+1}}\right)$.

(iii) *Ellipse and Hyperbola.*—Exactly the same process is applicable to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. We can treat these together here and elsewhere by writing the equation in the form $\frac{x^2}{\alpha} + \frac{y^2}{\beta} = 1$, where $\alpha = a^2$, and $\beta = + b^2$ for the ellipse, $- b^2$ for the hyperbola (§ 73). The equation for intersections with $y = mx + n$ is—

$$\frac{x^2}{\alpha} + \frac{(mx+n)^2}{\beta} = 1,$$

that is, $(\alpha m^2 + \beta)x^2 + 2\alpha mnx + \alpha(n^2 - \beta) = 0$.

The roots of this are equal if

$$\begin{aligned} \alpha^2m^2n^2 &= (\alpha m^2 + \beta)\alpha(n^2 - \beta), \\ \alpha m^2n^2 &= \alpha m^2n^2 + \beta n^2 - \beta(\alpha m^2 + \beta), \\ \beta n^2 &= \beta(\alpha m^2 + \beta), \\ n^2 &= \alpha m^2 + \beta. \end{aligned}$$

Hence $n = \pm\sqrt{\alpha m^2 + \beta}$, and there are two tangents with any given slope m, namely, $y = mx \pm \sqrt{\alpha m^2 + \beta}$. If we take $n = +\sqrt{\alpha m^2 + \beta}$, the equation for intersections becomes—

$$(\alpha m^2 + \beta)x^2 + 2\alpha m\sqrt{\alpha m^2 + \beta}\, x + \alpha^2m^2 = 0,$$

that is, $$(\sqrt{\alpha m^2 + \beta}\, x + \alpha m)^2 = 0.$$

Hence $x = -\dfrac{am}{\sqrt{am^2 + \beta}}$, $y = \dfrac{\beta}{\sqrt{am^2 + \beta}}$. Similarly from the value $-\sqrt{am^2 + \beta}$ for n we find—

$$x = +\frac{am}{\sqrt{am^2 + \beta}},\ y = -\frac{\beta}{\sqrt{am^2 + \beta}}.$$

Thus for all values of m, $y = mx + \sqrt{a^2m^2 + b^2}$ is a tangent to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at the point $\left(-\dfrac{a^2m}{\sqrt{a^2m^2 + b^2}}, \dfrac{b^2}{\sqrt{a^2m^2 + b^2}}\right)$, and $y = mx - \sqrt{a^2m^2 + b^2}$ is a tangent at $\left(\dfrac{a^2m}{\sqrt{a^2m^2 + b^2}}, \dfrac{-b^2}{\sqrt{a^2m^2 + b^2}}\right)$; and for all values of m, $y = mx + \sqrt{a^2m^2 - b^2}$ is a tangent to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ at the point $\left(\dfrac{-a^2m}{\sqrt{a^2m^2 - b^2}}, \dfrac{-b^2}{\sqrt{a^2m^2 - b^2}}\right)$, and $y = mx - \sqrt{a^2m^2 - b^2}$ is a tangent at $\left(\dfrac{a^2m}{\sqrt{a^2m^2 - b^2}}, \dfrac{b^2}{a^2m^2 - b^2}\right)$.

The conclusion arrived at, that there are two tangents to the circle, ellipse, or hyperbola, in any given direction, but only one to the parabola, agrees with our knowledge of the form of these curves.

EXAMPLES.

1. Find tangents with slope $\frac{3}{4}$ to the curves $y^2 = 12x$, $x^2 + y^2 = 25$, $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$, $x^2 - y^2 = 1$. Find the points of contact of these tangents.

2. Find a tangent to $y^2 = 4px$ that shall make equal intercepts on the axes; find the point of contact.

3. Find the equation of the tangent perpendicular to the tangent in (2). Where do these two tangents intersect?

4. Find the tangents to $\frac{x^2}{\alpha} + \frac{y^2}{\beta} = 1$ that make equal intercepts on the axes; and find their points of contact.

5. Find the coordinates of the point of intersection of two perpendicular tangents to $y^2 = 4px$. $\left(\text{Use slopes } m,\ -\frac{1}{m}.\right)$

6. Find the locus of the point of intersection of perpendicular tangents to a parabola.

7. Find the coordinates of the foot of the perpendicular from the focus of a parabola to a tangent.

8. Draw the lines $y = mx + \frac{p}{m}$ (with any convenient length for p), for $m = \pm 1,\ \pm 2,\ \pm 3,\ \pm 4$. Mark on each line the point $\left(\frac{p}{m^2}, \frac{2p}{m}\right)$.

80. It has been shown that for any given slope m there is one tangent to the parabola $y^2 = 4px$, namely $y = mx + \frac{p}{m}$. For all real values of m this is real.

The tangent to an ellipse is $y = mx \pm \sqrt{a^2m^2 + b^2}$. The quantity under the radical sign is positive for all real values of m, hence the two tangents are real; that is, in every real direction there are two real tangents to an ellipse.

The tangent to a hyperbola, $y = mx \pm \sqrt{a^2m^2 - b^2}$, is real if $a^2m^2 - b^2$ is positive, that is, if $m^2 > \frac{b^2}{a^2}$; but if $m^2 < \frac{b^2}{a^2}$, $a^2m^2 - b^2$ is negative, and the tangent is imaginary. That is, the two tangents to a hyperbola for a given real direction are not necessarily real.

If $m^2 = \frac{b^2}{a^2}$, the two tangents to the hyperbola coincide,

for $a^2m^2 - b^2 = 0$, hence both $y = mx + \sqrt{a^2m^2 - b^2}$ and $y = mx - \sqrt{a^2m^2 - b^2}$ reduce to $y = mx$. That is, for the slope $+\frac{b}{a}$, as also for the slope $-\frac{b}{a}$, there are not two distinct tangents; the two are represented by the one line $y = \frac{b}{a}x$, or $y = -\frac{b}{a}x$. In this case the coordinates of the point of contact are infinite. Each of the lines $\frac{x}{a} \pm \frac{y}{b} = 0$ is a tangent of a special character, the point of contact lying at infinity. These lines are called the asymptotes of the curve.

Definition of an asymptote.—An asymptote is a tangent whose point of contact is at infinity, while the line itself does not lie entirely at infinity.

The hyperbola has two real asymptotes, $\frac{x}{a} \pm \frac{y}{b} = 0$, and no others. For the coordinates of the point of contact of a tangent are infinite only if $a^2m^2 - b^2 = 0$.

The coordinates of the point of contact of a tangent to an ellipse are $\frac{\mp a^2m}{\sqrt{a^2m^2 + b^2}}, \frac{\pm b^2}{\sqrt{a^2m^2 + b^2}}$; these are infinite only if $a^2m^2 + b^2 = 0$, that is, only for two imaginary values of m. Hence the ellipse has no real asymptotes, but it has two imaginary asymptotes.

The point of contact of $y = mx + \frac{p}{m}$, a tangent to the parabola $y^2 = 4px$, is $\left(\frac{p}{m^2}, \frac{2p}{m}\right)$. These coordinates are infinite only if $m = 0$; but then the tangent is $y = 0 . x + \frac{p}{0}$, that is, $y = \infty$, which lies entirely at infinity. Hence

this tangent is not an asymptote. The parabola has therefore no asymptotes.

To recapitulate. The parabola has no asymptotes, real or imaginary, but it has a tangent that lies entirely at infinity; the ellipse has two imaginary asymptotes, whose combined equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$; the hyperbola has two real asymptotes, $\frac{x}{a} \pm \frac{y}{b} = 0$, whose combined equation is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$.

81. In tracing a hyperbola, it is generally convenient to draw the asymptotes, the lines $\frac{x}{a} \pm \frac{y}{b} = 0$. The simplest way is to mark the points $(a, \pm b)$; the lines joining these to the origin are the asymptotes. (That is, the asymptotes are the diagonals of the rectangle used in Chapter V, § 75.)

Any line through the origin, $y = mx$, meets the curve where $\frac{x^2}{a^2} - \frac{m^2x^2}{b^2} = 1$, that is, where $x^2 = \frac{a^2b^2}{b^2 - a^2m^2}$. Hence the two points are real if $m^2 < \frac{b^2}{a^2}$, imaginary if $m^2 > \frac{b^2}{a^2}$. This shows that the curve lies entirely in one pair of angles formed by the asymptotes. It will be shown in Chapter IX. (§ 127) that the curve continually approaches the asymptotes (Fig. 54). If we take this fact for granted at present, it is clear why the curve has no (real) tangents parallel to such a line as CH, for which $m < \frac{b}{a}$. Lines through C parallel to the real tangents lie in the

other pair of angles formed by the asymptotes, and have a slope m, which if positive is greater than $\frac{b}{a}$.

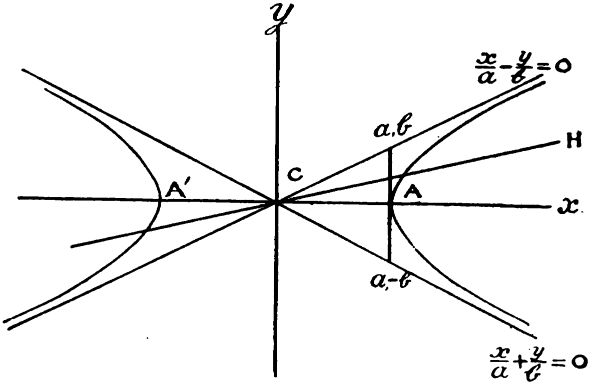


Fig. 54.

82. The following examples of the use of the formulæ proved above should be carefully noticed.

Example i.—Find the tangents to $y^2 = 4x$ from $\left(-\frac{2}{3}, \frac{5}{3}\right)$.

Any tangent has an equation,

$$y = mx + \frac{1}{m};$$

this passes through $\left(-\frac{2}{3}, \frac{5}{3}\right)$ if

$$\frac{5}{3} = -\frac{2}{3}m + \frac{1}{m}.$$

The slope m must therefore be chosen to satisfy this equation, that is—

$$2m^2 + 5m - 3 = 0,$$

$$(2m - 1)(m + 3) = 0;$$

hence $$m = \frac{1}{2} \text{ or } -3.$$

The tangents with these slopes are—

$$y = \frac{1}{2}x + 2, \quad y = -3x - \frac{1}{3},$$

that is, $x - 2y + 4 = 0$ and $9x + 3y + 1 = 0$,

and these meet at the given point $\left(-\frac{2}{3}, \frac{5}{3}\right)$.

Example ii.—Find the tangents to $\frac{x^2}{25} + \frac{y^2}{9} = 1$ from $\left(-15, -7\right)$.

Any tangent to this ellipse has an equation—

$$y = mx \pm \sqrt{25m^2 + 9}.$$

This passes through $(-15, -7)$ if

$$-7 = -15m \pm \sqrt{25m^2 + 9}.$$

The equation for the slope m is therefore—

$$\begin{aligned} 15m - 7 &= \pm \sqrt{25m^2 + 9}, \\ \text{that is,} \quad 225m^2 - 210m + 49 &= 25m^2 + 9, \\ 200m^2 - 210m + 40 &= 0, \\ 20m^2 - 21m + 4 &= 0, \\ (5m - 4)(4m - 1) &= 0. \end{aligned}$$

Hence $$m = \frac{4}{5} \text{ or } \frac{1}{4}.$$

Now there are *two* tangents with any specified slope; only one of these can pass through the point. If $m = \frac{4}{5}$, we have to decide whether the required tangent is

$$y = \frac{4}{5}x + \sqrt{25 \cdot \frac{16}{25} + 9}, \text{ that is, } y = \frac{4}{5}x + 5,$$

or $$y = \frac{4}{5}x - \sqrt{25 \cdot \frac{16}{25} + 9}, \text{ that is, } y = \frac{4}{5}x - 5.$$

We can tell by trial: $-15, -7$ do satisfy the first of these equations, they do not satisfy the second. Better still, we can use the formula for a line with slope m, through the point (x', y'), namely,

$$y - y' = m(x - x').$$

In this example, therefore, the required tangents are—

$$y + 7 = \frac{4}{5}(x + 15), \text{ that is, } 4x - 5y + 25 = 0,$$

and $$y + 7 = \frac{1}{4}(x + 15), \text{ that is, } x - 4y - 13 = 0.$$

EXAMPLES.

1. Find the tangents to $y^2 = 4x$ from the points $(2, 3)$, $(6, 5)$, $(-3, 2)$. Find also the points of contact.

2. Write down the equations of the normals to $y^2 = 4x$ at the points of contact of the tangents found in (1).

3. Find the tangents to $\frac{x^2}{25} + \frac{y^2}{16} = 1$ from $\left(7, \frac{4}{5}\right)$, and from $\left(1, \frac{28}{5}\right)$. Find also the points of contact.

4. Write down the equations of the normals to $\frac{x^2}{25} + \frac{y^2}{16} = 1$ at the points of contact of the tangents found in (3).

5. Find the tangents to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ from the points $\left(\frac{7a}{13}, \frac{17b}{13}\right)$, $\left(\frac{a}{5}, \frac{7b}{5}\right)$.

83. The process applied to numerical examples in the last section is the general process for finding the slopes of tangents from a point (x_1, y_1) to a conic.

(i) *Parabola.*—The tangent to $y^2 = 4px$ is—

$$y = mx + \frac{p}{m};$$

this passes through the given point (x_1, y_1) if

$$y_1 = mx_1 + \frac{p}{m}.$$

The only quantity at our disposal is m; this must there-

fore be chosen to satisfy this equation. Hence for the determination of m we have the equation—

$$y_1 m = x_1 m^2 + p,$$

that is, $$x_1 m^2 - y_1 m + p = 0,$$

a quadratic. Hence there are two values for m, call these m_1, m_2; through the point (x_1, y_1) there pass two tangents to the parabola, namely—

$$y = m_1 x + \frac{p}{m_1},\ y = m_2 x + \frac{p}{m_2}.$$

In particular, the two tangents will be perpendicular if $m_1 m_2 = -1$; now the quadratic for m shows that $m_1 m_2 = \frac{p}{x_1}$, hence if $\frac{p}{x_1} = -1$, that is, if $x_1 + p = 0$, the tangents from (x_1, y_1) are at right angles. But if $x_1 + p = 0$, the point (x_1, y_1) lies on the directrix; hence the result obtained can be stated in the form—tangents to a parabola from any point on the directrix are at right angles; or, the locus of the intersection of perpendicular tangents to a parabola is the directrix.

(ii) *Ellipse and Hyperbola.*—Similarly the slopes of tangents from (x_1, y_1) to an ellipse or hyperbola, $\frac{x^2}{a} + \frac{y^2}{\beta} = 1$, are determined by a quadratic equation. Any tangent is $y = mx \pm \sqrt{am^2 + \beta}$; this passes through (x_1, y_1) if $y_1 = mx_1 \pm \sqrt{am^2 + \beta}$. The only quantity at our disposal is m; this is therefore an equation for m, and must be arranged accordingly. The equation is—

$$mx_1 - y_1 = \pm\sqrt{am^2 + \beta},$$

that is, $$x_1^2 m^2 - 2x_1 y_1 m + y_1^2 = am^2 + \beta,$$

$$(x_1^2 - a)m^2 - 2x_1 y_1 m + y_1^2 - \beta = 0.$$

If m_1, m_2 are the roots of this quadratic, the two tangents from (x_1, y_1) are—

$$y - y_1 = m_1(x - x_1), \quad y - y_1 = m_2(x - x_1).$$

In particular, the two tangents will be perpendicular if $m_1 m_2 = -1$, that is, if

$$\frac{y_1^2 - \beta}{x_1^2 - \alpha} = -1,$$

which gives $$x_1^2 + y_1^2 = \alpha + \beta.$$

Hence the tangents from (x_1, y_1) are at right angles if x_1, y_1 satisfy the equation $x_1^2 + y_1^2 = \alpha + \beta$, that is, if (x_1, y_1) is a point on the curve $x^2 + y^2 = \alpha + \beta$, which is a circle, whose centre is at the centre of the conic. Hence the result obtained is—the locus of the intersection of perpendicular tangents to an ellipse or hyperbola is the circle $x^2 + y^2 = \alpha + \beta$.

For the ellipse, the circle is $x^2 + y^2 = a^2 + b^2$; for the hyperbola, it is $x^2 + y^2 = a^2 - b^2$.

Note.—This circle is sometimes called the director circle of the conic, sometimes the orthocycle.

EXAMPLES.

1. Form the equation of a tangent to $\frac{x^2}{9} + \frac{y^2}{4} = 1$ with slope $\frac{1}{2}$; also of a perpendicular tangent. Find the coordinates of the point of intersection, and verify that it lies on the director circle.

2. Show that a hyperbola has no real perpendicular tangents if $a < b$, and only one pair if $a = b$.

3. The distance from the centre of $\frac{x^2}{7} - \frac{y^2}{2} = 1$ to a certain tangent is twice the distance from the centre to the perpendicular

tangent. Find the two tangents, and the distance of each from the centre.

4. The distance from the centre of $\frac{x^2}{36} + \frac{y^2}{9} = 1$ to a tangent is 5. Find the slope of the tangent. Find also the distance from the centre to the perpendicular tangent.

84. In the last section it was shown that two tangents to a conic pass through any point. This conclusion was drawn from the fact that the slope of the tangent is determined by a quadratic equation. If the roots of the quadratic are real, the two tangents are real; if however the roots of the quadratic are imaginary, the two tangents are imaginary. For example, the slopes of the tangents from the point (4, 2) to the parabola $y^2 = 4x$ are determined by—

$$2 = 4m + \frac{1}{m},$$

that is, $4m^2 - 2m + 1 = 0$.

Since the roots of this equation are imaginary, there are no real tangents from (4, 2).

EXAMPLE.

Illustrate this by a careful diagram.

The equation that determines the slopes of the tangents from (x_1, y_1) to the parabola $y^2 = 4px$ is—

$$x_1m^2 - y_1m + p = 0.$$

The roots of this are real if $y_1^2 - 4px_1$ is positive, imaginary if $y_1^2 - 4px_1$ is negative. If we write the equation of the curve as—

$$y^2 - 4px = 0,$$

and represent the expression $y^2 - 4px$ by the single letter u (§ 42), this result takes a convenient form; the tangents from a point P are real if u_P is positive, imaginary if u_P is negative. In Chap. III. (§ 42) it was shown that if u is an expression of the first degree, the straight line $u = 0$ divides the plane into two regions, in one of which the expression u is positive, while in the other it is negative. It will now be proved that corresponding facts hold for the parabola, ellipse (including circle) and hyperbola.

85. (i) *Parabola.*—If u be written for $y^2 - 4px$, the equation of the curve is $u = 0$. It is to be proved that u_P is positive if P is on the convex side of the parabola, negative if P is on the concave side.

Let P be (x_1, y_1), then u_P is $y_1^2 - 4px_1$.

If $\qquad x_1$ is negative,

$\qquad - 4px_1$ is positive,

hence $\qquad y_1^2 - 4px_1$ is positive,

that is, for a point P on the side of the axis of y that is remote from the parabola, u_P is positive (Fig. 55 (1)).

If x_1 is positive, draw NP, the ordinate of P, meeting the parabola at Q; let NQ $= y_2$. Since Q is on the curve, its coordinates (x_1, y_2) satisfy the equation—

$$y_2^2 - 4px_1 = 0.$$

Now $\qquad u_P = y_1^2 - 4px_1,$

hence, by subtraction,

$$u_P = y_1^2 - y_2^2.$$

If NP > NQ, that is, if $y_1 > y_2$, u_P is positive (Fig. 55 (2)).

If, however, NP < NQ, that is, if $y_1 < y_2$, u_P is negative (Fig. 55 (3)).

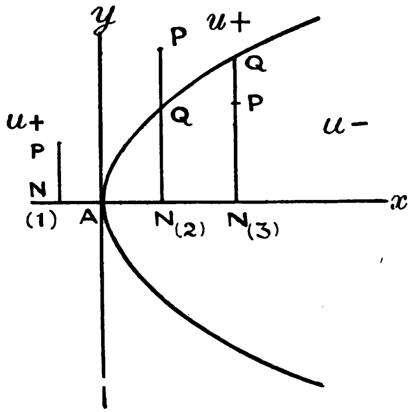


Fig. 55.

Thus for points P, such as those marked on the convex side of the parabola, u is positive; while for a point P on the concave side, u is negative. The parabola $u = 0$ divides the plane into two regions, in one of which the expression u is positive, while in the other it is negative.

(ii) *Ellipse.*—Here $u = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$.

If $$x_1^2 > a^2,$$

$$\frac{x_1^2}{a^2} - 1 \text{ is positive,}$$

hence $$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 \text{ is positive,}$$

that is, u_P is positive (Fig. 56 (1)).

If $x_1^2 < a^2$, the ordinate of P, NP, cuts the curve at the point Q, whose coordinates x_1, y_2 satisfy the equation of the curve, that is—

$$\frac{x_1^2}{a^2} + \frac{y_2^2}{b^2} - 1 = 0.$$

Now $$u_P = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1,$$

hence, by subtraction,

$$u_P = \frac{y_1^2 - y_2^2}{b^2}.$$

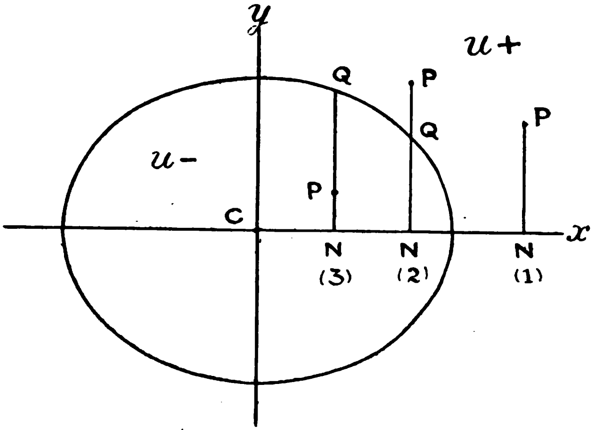


Fig. 56.

If $\quad$ NP > NQ (Fig. 56 (2)),

that is, if $\quad y_1 > y_2$,

u_P is positive;

but if $\quad$ NP < NQ (Fig. 56 (3)),

that is, if $\quad y_1 < y_2$,

u_P is negative.

Hence the expression u has different signs in the two regions into which the plane is divided by the ellipse; on the convex side of the curve, u is positive; while on the concave side, u is negative.

(iii) *Hyperbola.*—For the hyperbola, $u = \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1.$

If $$x_1^2 < a^2,$$
$$\frac{x_1^2}{a^2} - 1 \text{ is negative,}$$

hence $$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 \text{ is negative,}$$

that is, u_P is negative (Fig. 57 (1)).

If $x_1^2 > a^2$, NP, the ordinate of P, cuts the curve at the point Q, whose coordinates x_1, y_2 satisfy the equation—

$$\frac{x_1^2}{a^2} - \frac{y_2^2}{b^2} - 1 = 0.$$

Now $$u_P = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1,$$

hence, by subtraction,

$$u_P = \frac{-y_1^2 + y_2^2}{b^2}.$$

If $\text{NP} > \text{NQ}$ (Fig. 57 (2)),

that is, if $y_1 > y_2$,

u_P is negative;

but if $\text{NP} < \text{NQ}$ (Fig. 57 (3)),

that is, if $y_1 < y_2$,

u_P is positive.

Hence the expression u is negative on the convex side

of the curve, while in the two regions on the concave side it is positive.

Thus we see that any conic divides the plane into regions, which are distinguished by the sign of the expression u. A region in which u is positive is separated

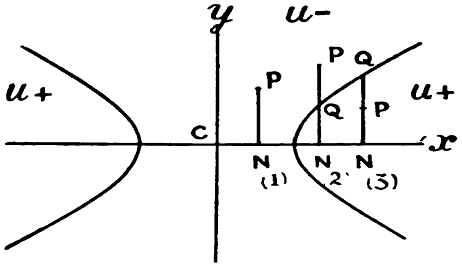


Fig. 57.

from a region in which u is negative by a part of the curve $u = 0$.

Definition of the power of a point.—The value of an expression u at any point is called the power of that point with respect to the curve $u = 0$.

Note.—Care must be exercised in determining the signs of u in the different regions. For example, if u be written for $-\frac{x^2}{a^2} + \frac{y^2}{b^2} + 1$, then u is positive on the convex side of the hyperbola $u = 0$, negative on the concave side. It is useful to remember that the centre of an ellipse is on the concave side, while the centre of a hyperbola is on the convex side.

86. The result of § 84, that the tangents from a point P to the parabola are real or imaginary according as u_P is positive or negative (or, according as the power of P is

positive or negative), is equivalent to the geometrical statement—two real tangents to a parabola pass through any point on the convex side of the curve, and two imaginary tangents pass through any point on the concave side.

This geometrical statement is true also for the ellipse and hyperbola; the equation which determines the slopes of the tangents from (x_1, y_1) to $\frac{x^2}{a} + \frac{y^2}{\beta} = 1$, namely,

$$(x_1^2 - a)m^2 - 2x_1y_1m + (y_1^2 - \beta) = 0,$$

has real roots if $x_1^2y_1^2 - (x_1^2 - a)(y_1^2 - \beta)$ is positive, that is, if $\beta x_1^2 + ay_1^2 - a\beta$ is positive.

$$\begin{aligned}\text{Now } \beta x_1^2 + ay_1^2 - a\beta & \\ &= a\beta\left(\frac{x_1^2}{a} + \frac{y_1^2}{\beta} - 1\right) \\ &= a\beta u_P, \text{ where P is the point } (x_1, y_1).\end{aligned}$$

The product $a\beta$ is positive for the ellipse, negative for the hyperbola; hence the tangents from P to an ellipse are real if u_P is positive, that is, if P lies on the convex side, and the tangents from P to a hyperbola are real if u_P is negative, that is, if P lies on the convex side.

EXAMPLES.

1. On which side of $y^2 = 8x$, also of $x^2 - 10y = 0$, are the points (10, 2), (3, 3), (2, 9), (− 8, 3)?

2. Express algebraically that a point (x_1, y_1) is on the concave side of each of the parabolas in No. (1). Also that a point is on the convex side of each.

3. Are the tangents to $\frac{x^2}{25} + \frac{y^2}{16} = 1$ from the points (4, 4), (4, 3), (4, 2), real or imaginary?

4. Express algebraically that the tangents from a point to both curves $\frac{x^2}{25}+\frac{y^2}{16}=1$, $\frac{x^2}{16}-\frac{y^2}{9}=1$ are imaginary.

5. Show that there is no point from which the tangents to both curves $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are imaginary.

87. It has been shown that the line $y = mx + n$ is a tangent to the parabola $y^2 = 4px$ if $mn = p$, and to the central conic $\frac{x^2}{a}+\frac{y^2}{\beta}=1$ if $n^2 = am^2 + \beta$. Since the coordinates of the line are expressible in terms of m, n these conditions imposed on m, n are equivalent to equations to be satisfied by ξ, η, the coordinates of a tangent to the conic. The line $y = mx + n$ can be written—

$$\frac{m}{n}x - \frac{1}{n}y + 1 = 0.$$

Hence $$\xi = \frac{m}{n}, \quad \eta = -\frac{1}{n},$$

that is, $$m = -\frac{\xi}{\eta},\ n = -\frac{1}{\eta}.$$

The condition that the line (ξ, η) be a tangent to the parabola $y^2 = 4px$ is therefore—

$$\frac{\xi}{\eta^2} = p,$$

that is, $$\eta^2 = \frac{1}{p}\cdot\xi;$$

and the condition that the line be a tangent to $\frac{x^2}{a}+\frac{y^2}{\beta}=1$ is—

$$\frac{1}{\eta^2} = a\frac{\xi^2}{\eta^2} + \beta,$$

that is, $$a\xi^2 + \beta\eta^2 = 1.$$

Instead of deducing these equations from the condition that the line $y = mx + n$ be a tangent, it is better to give an independent proof, by finding the condition that the line $\xi x + \eta y + 1 = 0$ be a tangent to the conic. The most symmetrical way is to form the equation of the lines that join the origin to the common points P, Q (§ 64), and express that these lines, OP, OQ are indistinguishable; this will make the points P, Q indistinguishable, and then the line PQ, that is, $\xi x + \eta y + 1 = 0$, will be a tangent. If the conic be the parabola, $y^2 = 4px$, the lines OP, OQ are $y^2 = 4px \times -(\xi x + \eta y)$, that is—

$$4p\xi . x^2 + 4p\eta . xy + y^2 = 0.$$

These are indistinguishable if $(2p\eta)^2 = 4p\xi$, that is, if $\eta^2 = \frac{1}{p} . \xi$.

For the central conic, $\frac{x^2}{a} + \frac{y^2}{\beta} = 1$, the lines OP, OQ are—

$$\frac{x^2}{a} + \frac{y^2}{\beta} = (\xi x + \eta y)^2,$$

that is,

$$\left(\xi^2 - \frac{1}{a}\right)x^2 + 2\xi\eta xy + \left(\eta^2 - \frac{1}{\beta}\right)y^2 = 0.$$

These are indistinguishable if $(\xi\eta)^2 = \left(\xi^2 - \frac{1}{a}\right)\left(\eta^2 - \frac{1}{\beta}\right)$, that is, if $a\xi^2 + \beta\eta^2 = 1$.

88. These equations have a very important significance. The tangents to a conic form a system of lines; for every line of the system a certain equation is satisfied by the coordinates. This is the equation of the envelope of a

line, supposed to move in such a manner as to represent in succession all the lines of the system. (Compare § 53, Chapter IV.). The tangents to the parabola $y^2 = 4px$ satisfy the equation $\eta^2 = \frac{1}{p}\xi$; this is the *line-equation of the parabola.* The tangents to the central conic $\frac{x^2}{\alpha} + \frac{y^2}{\beta} = 1$ satisfy the equation $\alpha\xi^2 + \beta\eta^2 = 1$; this is the *line-equation of the central conic.*

Note.—The line-equation is sometimes called the tangential equation of the curve.

Since the line-equation of a conic is of the second degree, the conic is a curve of the second *class*[1] (§ 76). This has already been proved in another manner; it was shown in § 83 that two tangents pass through any point.

89. The problems of § 82, on finding tangents from a point, are solved neatly by means of the line-equations.

Example i.—Find the tangents to $y^2 = 4x$ from $\left(-\frac{2}{3}, \frac{5}{3}\right)$.

The line $\xi x + \eta y + 1 = 0$ is a tangent if $\eta^2 = \xi$;

it passes through $\left(-\frac{2}{3}, \frac{5}{3}\right)$ if $-\frac{2}{3}\xi + \frac{5}{3}\eta + 1 = 0$,

that is, if $$+ 2\xi - 5\eta - 3 = 0.$$

Hence
$$+ 2\eta^2 - 5\eta - 3 = 0,$$
$$(2\eta + 1)(\eta - 3) = 0,$$
$$\eta = -\frac{1}{2}, \text{ which gives } \xi = \frac{1}{4};$$
or
$$\eta = 3, \quad \text{which gives } \xi = 9.$$

---

[1] For curves of order higher than the second, the class is usually different from the order.

The two tangents are therefore—

$$\frac{1}{4}x - \frac{1}{2}y + 1 = 0,$$

that is,
$$x - 2y + 4 = 0,$$
and
$$9x + 3y + 1 = 0.$$

Example ii.—Find the tangents to $\frac{x^2}{25} + \frac{y^2}{9} = 1$ from $(-15, \quad 7)$.

The coordinates are determined by—

$$25\xi^2 + 9\eta^2 = 1,$$
$$15\xi + 7\eta - 1 = 0.$$

The equation for η is therefore—

$$(7\eta - 1)^2 + 9(9\eta^2 - 1) = 0,$$
$$130\eta^2 - 14\eta - 8 = 0,$$
$$65\eta^2 - 7\eta - 4 = 0,$$
$$(5\eta + 1)(13\eta - 4) = 0.$$

Hence
$$\eta = -\frac{1}{5}, \quad \xi = \frac{4}{25};$$
or
$$\eta = \frac{4}{13}, \quad \xi = -\frac{1}{13}.$$

The required tangents are therefore—

$$\frac{4}{25}x - \frac{1}{5}y + 1 = 0, \text{ that is, } 4x - 5y + 25 = 0,$$

and
$$-\frac{1}{13}x + \frac{4}{13}y + 1 = 0, \quad \text{that is, } x - 4y - 13 = 0.$$

Examples.

1. Find the tangents to $y^2 = 4x$ from the points $(2, 3)$, $(-3, 2)$ by means of the line-equation of the curve.

2. Find the line-equation of $\frac{x^2}{25} + \frac{y^2}{16} = 1$. Apply this to find the tangents from $\left(7, \frac{4}{5}\right)$, and from $\left(1, \frac{28}{5}\right)$.

3. Find the tangents to $a^2\xi^2 + b^2\eta^2 = 1$ from the points $\left(\frac{7a}{13}, \frac{17b}{13}\right)$, $\left(\frac{a}{5}, \frac{7b}{5}\right)$

4. Find the four common tangents to $\frac{x^2}{16} + \frac{y^2}{9} = 1$, $\frac{x^2}{30} - \frac{y^2}{5} = 1$.
Show that these are also tangents to $\frac{x^2}{20} + \frac{y^2}{5} = 1$.

5. Find the line-equation of the curve $2xy = k^2$.

90. We have shown that there are two tangents, real or imaginary, from a point (x_1, y_1). The slopes of these tangents are given by a certain quadratic equation; and if m be the slope of either tangent, its equation is—

$$y - y_1 = m(x - x_1).$$

Hence by eliminating m from these two equations, we obtain an equation that connects x, y, the coordinates of any point on either tangent, with x_1, y_1, the coordinates of the point through which the two tangents pass; that is, we obtain the equation of the pair of tangents.

(i) *Parabola.*—A tangent is $y = mx + \frac{p}{m}$;

this passes through (x_1, y_1) if $y_1 = mx_1 + \frac{p}{m}$,

this is therefore the equation for m.

Again, for a point (x, y) on the tangent—

$$y - y_1 = m(x - x_1),$$

hence $$m = \frac{y - y_1}{x - x_1}.$$

Eliminating m by direct substitution we have—

$$y_1 = \frac{y - y_1}{x - x_1} x_1 + p \frac{x - x_1}{y - y_1},$$

that is, $$\frac{xy_1 - x_1 y}{x - x_1} = p \frac{x - x_1}{y - y_1}.$$

Hence $p(x - x_1)^2 - (y - y_1)(xy_1 - x_1y) = 0$ is the equation of the pair of tangents from (x_1, y_1) to $y^2 = 4px$.

(ii) *Central conic.*—The slopes of the tangents through (x_1, y_1) are given by—

$$y_1 - mx_1 = \pm\sqrt{am^2 + \beta},$$

and for any point (x, y) on a line through (x_1, y_1) with slope m,

$$m = \frac{y - y_1}{x - x_1}.$$

Eliminating m, we obtain—

$$y_1 - \frac{y - y_1}{x - x_1}x_1 = \pm\sqrt{a\left(\frac{y - y_1}{x - x_1}\right)^2 + \beta},$$

that is, $$xy_1 - x_1y = \pm\sqrt{a(y - y_1)^2 + \beta(x - x_1)^2}.$$

Hence $(xy_1 - x_1y)^2 = \beta(x - x_1)^2 + a(y - y_1)^2$ is the equation of the pair of tangents from (x_1, y_1) to $\frac{x^2}{a} + \frac{y^2}{\beta} = 1$.

EXAMPLES.

1. Form the equation of the line that joins the points of contact of perpendicular tangents (one with slope m) to a parabola, and show that it passes through the focus. Show that the inclination of this line is twice the inclination of one of the tangents.

2. Find the locus of the foot of the perpendicular from the focus of a parabola to a tangent.

3. Find the locus of the foot of the perpendicular from the vertex of a parabola to a tangent.

4. Find the coordinates of the point of intersection of two tangents to a parabola, of slopes m_1, m_2. Find also the equation of the chord that joins the points of contact. What condition is satisfied by m_1, m_2 if the slope of this chord has a constant value? In this case find the locus of the point of intersection of the two tangents.

5. Three tangents to a parabola, with slopes m_1, m_2, m_3, form a

triangle. Find the equation of the lines through the vertices perpendicular to the opposite sides. Prove that these three lines meet at a point which lies on the directrix.

6. Find the slopes of the two tangents to a parabola from the focus. Show that the points of contact lie on the directrix.

7. Find the locus of the point of intersection of two tangents to a parabola which meet at a constant angle.

8. The sum of the inclinations of two tangents to a parabola is constant. Prove that the locus of their point of intersection is a straight line through the focus. What is the inclination of this line?

9. Prove that the chord of contact of the two tangents in No. 8 meets the directrix at a fixed point.

10. Find the locus of the foot of the perpendicular from a focus of a central conic to a tangent.

11. Find the locus of the foot of the perpendicular from the centre of an ellipse or hyperbola to a tangent.

12. The slopes of two tangents to an ellipse are connected by the relation $m_1m_2 = -\frac{b^2}{a^2}$; find the locus of their point of intersection.

13. State and solve the problem that corresponds to No. 12 for the case of the hyperbola.

14. Find the slopes of the two tangents to a central conic from a focus. Show that the points of contact lie on the corresponding directrix.

15. Find the condition that the line $y = mx + n$ shall meet the parabola $y^2 = 4px$ in real points. Show that if p is positive, the condition is $mn < p$.

16. Find the condition that the line (ξ, η) shall meet the parabola $\eta^2 = \frac{1}{p}\xi$ in real points (p is supposed to be positive).

17. Find the condition that $y = mx + n$ shall meet $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in real points.

18. Find the condition that the line (ξ, η) shall meet $\alpha\xi^2 + \beta\eta^2 = 1$ in real points.

19. Prove that if the tangents from (x_1, y_1) to an ellipse are real, the line whose coordinates are $-\frac{x_1}{a^2}$, $-\frac{y_1}{b^2}$ meets the ellipse in real points.

20. Two central conics whose axes lie along the same straight lines have the same director circle. Prove that the common tangents are the sides of a square.

21. Prove that the product of the distances from the two foci of an ellipse to a tangent is b^2. Find the product for a hyperbola.

22. Find the tangents to $\dfrac{x^2}{100} + \dfrac{y^2}{25} = 1$ from the point (2, 7).

CHAPTER VII

TANGENT AT A POINT. POLAR PROPERTIES

91. THE tangent has been defined (§ 78) as the limiting position of a chord PQ, when Q moves along the curve towards P. Hence to find the equation of the tangent at a point P on any curve, take another point Q on the curve, write down the equation of the line PQ, and determine the form this equation assumes when Q, while remaining on the curve, approaches P indefinitely. We shall now find the tangents to various curves by this method.

(i) *Parabola.*—To find the tangent at P, (x_1, y_1) to the parabola $y^2 = 4px$, take a point Q, (x_2, y_2). The equation of PQ is—

$$y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1);$$

the slope is $\frac{y_1 - y_2}{x_1 - x_2}$. If we now make (x_2, y_2) approach (x_1, y_1), both numerator and denominator of this fraction become zero. We avoid the apparent difficulty by making use of the conditions that P and Q are points on the parabola, which are expressed by the equations—

$$y_1^2 = 4px_1,$$
$$y_2^2 = 4px_2.$$

Since the differences $y_1 - y_2$, $x_1 - x_2$ are required, we subtract, thus obtaining—

$$y_1^2 - y_2^2 = 4p(x_1 - x_2),$$

that is, $(y_1 - y_2)(y_1 + y_2) = 4p(x_1 - x_2)$.

Hence $$\frac{y_1 - y_2}{x_1 - x_2} = \frac{4p}{y_1 + y_2},$$

that is, the slope of a chord of a parabola is $\dfrac{4p}{y_1 + y_2}$. The equation of the chord is therefore—

$$y - y_1 = \frac{4p}{y_1 + y_2}(x - x_1).$$

No difficulty now arises if we make Q approach P, by writing $x_2 = x_1$, $y_2 = y_1$. The slope becomes $\dfrac{4p}{2y_1}$, that is, $\dfrac{2p}{y_1}$, and the equation of the tangent is therefore—

$$y - y_1 = \frac{2p}{y_1}(x - x_1),$$

that is, $$yy_1 - y_1^2 = 2px - 2px_1.$$

Since $y_1^2 = 4px_1$, this equation reduces to—

$$yy_1 = 2p(x + x_1).$$

(ii) *Central conics.*—To find the tangent at P to $\dfrac{x^2}{a} + \dfrac{y^2}{\beta} = 1$, take a second point $Q(x_2, y_2)$. The equation of the chord PQ is—

$$y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1),$$

the slope is $\dfrac{y_1 - y_2}{x_1 - x_2}$. Before making Q approach P,

express that both points are on the curve; this gives two equations—

$$\frac{x_1^2}{a}+\frac{y_1^2}{\beta}=1,$$

$$\frac{x_2^2}{a}+\frac{y_2^2}{\beta}=1.$$

To bring in differences, subtract—

$$\frac{x_1^2-x_2^2}{a}+\frac{y_1^2-y_2^2}{\beta}=0,$$

that is, $$\frac{(x_1-x_2)(x_1+x_2)}{a}+\frac{(y_1-y_2)(y_1+y_2)}{\beta}=0.$$

Hence $$\frac{y_1-y_2}{x_1-x_2}=-\frac{\beta}{y_1+y_2}\cdot\frac{x_1+x_2}{a}=-\frac{\beta(x_1+x_2)}{a(y_1+y_2)},$$

that is, the slope of the chord is $-\frac{\beta(x_1+x_2)}{a(y_1+y_2)}$, and the equation of the chord is—

$$y-y_1=-\frac{\beta(x_1+x_2)}{a(y_1+y_2)}(x-x_1).$$

To obtain the tangent, make Q approach P indefinitely by writing $x_2=x_1$, $y_2=y_1$. This causes no difficulty; the slope becomes $-\frac{2\beta x_1}{2ay_1}$, that is, $-\frac{\beta x_1}{ay_1}$, and the equation of the tangent is therefore—

$$y-y_1=-\frac{\beta x_1}{ay_1}(x-x_1),$$

that is, $$\frac{x_1}{a}(x-x_1)+\frac{y_1}{\beta}(y-y_1)=0,$$

or $$\frac{xx_1}{a}+\frac{yy_1}{\beta}=\frac{x_1^2}{a}+\frac{y_1^2}{\beta}.$$

Since P is on the curve, $\dfrac{x_1^2}{\alpha} + \dfrac{y_1^2}{\beta} = 1$, and the equation of the tangent becomes—

$$\frac{xx_1}{\alpha} + \frac{yy_1}{\beta} = 1.$$

If the equation of the curve be given in the form $Ax^2 + By^2 + C = 0$, the equation of the tangent is $Axx_1 + Byy_1 + C = 0$.

Thus the tangent at a point (x_1, y_1) on the circle $x^2 + y^2 = r^2$ is $xx_1 + yy_1 = r^2$; at a point on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, the tangent is $\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = 1$; and at a point on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, the tangent is $\dfrac{xx_1}{a^2} - \dfrac{yy_1}{b^2} = 1$.

(iii) *Circle.*—The general equation of a circle is—

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

As before, the slope of the chord PQ is $\dfrac{y_1 - y_2}{x_1 - x_2}$, where

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0,$$
$$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0.$$

From these equations there follows by subtraction—

$$(x_1^2 - x_2^2) + (y_1^2 - y_2^2) + 2g(x_1 - x_2) + 2f(y_1 - y_2) = 0,$$

that is—

$$(x_1 - x_2)(x_1 + x_2 + 2g) + (y_1 - y_2)(y_1 + y_2 + 2f) = 0;$$

hence $$\frac{y_1 - y_2}{x_1 - x_2} = -\frac{x_1 + x_2 + 2g}{y_1 + y_2 + 2f},$$

and the equation of the chord becomes—

$$y - y_1 = -\frac{x_1 + x_2 + 2g}{y_1 + y_2 + 2f}(x - x_1).$$

If we now write $x_2 = x_1, y_2 = y_1$, we obtain the equation of the tangent—

$$y - y_1 = -\frac{x_1 + g}{y_1 + f}(x - x_1),$$

that is, $$(x_1 + g)(x - x_1) + (y_1 + f)(y - y_1) = 0.$$

This is—

$$(x_1 + g)x + (y_1 + f)y - (x_1^2 + y_1^2 + gx_1 + fy_1) = 0.$$

Now $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$; adding this to the equation above, we obtain—

$$(x_1 + g)x + (y_1 + f)y + gx_1 + fy_1 + c = 0,$$

which is usually written—

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

(iv) As another illustration of the method, take a curve of the third order—

$$y^2 = x^3.$$

Since $$y_1^2 = x_1^3,$$
and $$y_2^2 = x_2^3,$$
therefore $$y_1^2 - y_2^2 = x_1^3 - x_2^3,$$
that is, $$(y_1 - y_2)(y_1 + y_2) = (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2).$$

The slope, $\frac{y_1 - y_2}{x_1 - x_2}$, of a chord PQ is therefore—

$$\frac{x_1^2 + x_1x_2 + x_2^2}{y_1 + y_2},$$

and the equation of the chord is—

$$y - y_1 = \frac{x_1^2 + x_1x_2 + x_2^2}{y_1 + y_2}(x - x_1).$$

The slope of the tangent is $\frac{3x_1^2}{2y_1}$, and the equation of the tangent is consequently—

$$y - y_1 = \frac{3x_1^2}{2y_1}(x - x_1),$$

that is, $$2y_1(y - y_1) = 3x_1^2(x - x_1),$$

which can be written—

$$2yy_1 + y_1^2 = 3xx_1^2.$$

EXAMPLES.

1. Write down the equations of the tangents to the circle $x^2 + y^2 = 25$ at the points where it is met by the line $x + y = 7$.

2. Find the condition satisfied by the coordinates of two points (x_1, y_1), (x_2, y_2) on the parabola $y^2 = 4px$ if the tangents are perpendicular. Express this in terms of x_1, x_2.

3. Find the equation of the tangent at (x_1, y_1) to $y^2 = 4px + q$.

4. Find the equation of the tangent at (x_1, y_1) to $2xy = k$.

5. Find the tangents at the vertices of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

6. The tangent to an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_1, y_1) is perpendicular to the tangent at (x_2, y_2). Find the relation that connects the coordinates of the two points.

7. Show that if the sum of the ordinates of two points on a parabola is equal to the sum of the ordinates of two other points, the chord that joins the first two points is parallel to the chord that joins the other two points.

92. A slightly different way of writing the proof is generally more convenient for curves of order higher than the second; this may of course be employed also for curves of the second order. Instead of denoting the coordinates of Q by x_2, y_2, express them as $x_1 + \triangle x_1, y_1 + \triangle y_1$, so that $\triangle x_1, \triangle y_1$ stand for $x_2 - x_1, y_2 - y_1$, that is, for the

increments in x, y in passing from P to Q (Fig. 58). The slope of the chord PQ, which is $\frac{y_1 - y_2}{x_1 - x_2}$, becomes $\frac{\Delta y_1}{\Delta x_1}$, and the equation of the chord is therefore—

$$y - y_1 = \frac{\Delta y_1}{\Delta x_1}(x - x_1).$$

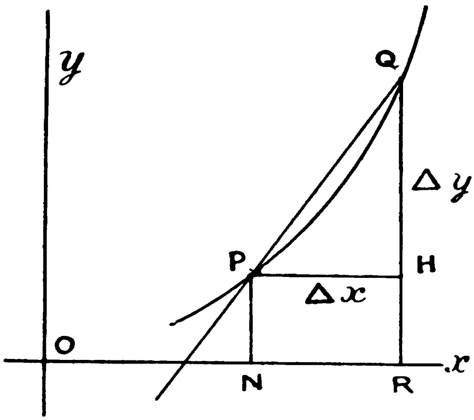


Fig. 58.

Example i.—The curve $y = x^3$.

Since P is on the curve,

$$y_1 = x_1^3.$$

Since Q is on the curve,

$$y_1 + \Delta y_1 = (x_1 + \Delta x_1)^3,$$
$$= x_1^3 + 3x_1^2 \Delta x_1 + 3x_1(\Delta x_1)^2 + (\Delta x_1)^3.$$

Hence, by subtraction,

$$\Delta y_1 = 3x_1^2 \Delta x_1 + 3x_1(\Delta x_1)^2 + (\Delta x_1)^3,$$

therefore $$\frac{\Delta y_1}{\Delta x_1} = 3x_1^2 + 3x_1(\Delta x_1) + (\Delta x_1)^2,$$

and the equation of the chord is—

$$y - y_1 = [3x_1^2 + 3x_1 \Delta x_1 + (\Delta x_1)^2](x - x_1).$$

To obtain the tangent, write $x_2 = x_1$, $y_2 = y_1$, that is, $\triangle x_1 = 0$, $\triangle y_1 = 0$. The equation is then—

$$y - y_1 = 3x_1^2(x - x_1),$$

which can be reduced to the form—

$$y + 2y_1 = 3x_1^2 x.$$

Example ii.—The curve $2xy = k$ is a hyperbola (§ 138).

Since P, Q are on the curve,

$$2x_1 y_1 = k,$$
$$2(x_1 + \triangle x_1)(y_1 + \triangle y_1) = k;$$

from these, by subtraction,

$$x_1 \triangle y_1 + y_1 \triangle x_1 + \triangle x_1 \cdot \triangle y_1 = 0.$$

This can be written in either of the forms—

$$(x_1 + \triangle x_1) \triangle y_1 + y_1 \cdot \triangle x_1 = 0,$$
$$x_1 \cdot \triangle y_1 + (y_1 + \triangle y_1) \triangle x_1 = 0;$$

hence the slope of a chord, $\frac{\triangle y_1}{\triangle x_1}$, can be expressed either as $-\frac{y_1}{x_1 + \triangle x_1}$ or as $-\frac{y_1 + \triangle y_1}{x_1}$. Whichever form is used, it reduces to $-\frac{y_1}{x_1}$ when Q is made to approach P indefinitely. Hence the equation of the tangent is—

$$y - y_1 = -\frac{y_1}{x_1}(x - x_1),$$

that is, $\quad y_1(x - x_1) + x_1(y - y_1) = 0,$

which in virtue of the relation $2x_1 y_1 = k$, reduces to—

$$y_1 x + x_1 y = k.$$

93. If the curve is $f(x, y) = 0$, the conditions that P, Q lie on it are $f(x_1, y_1) = 0$, $f(x_1 + \triangle x_1, y_1 + \triangle y_1) = 0$. On expanding the second of these, and subtracting the first, it will be found that every term left contains either $\triangle x_1$ or $\triangle y_1$ (some power, or product of powers) as in the

two examples worked out. The result can therefore be arranged in various ways in the form—

$\Delta x_1 \times$ a number of terms which may or may not involve $\Delta x_1, \Delta y_1$

$+ \Delta y_1 \times$ a number of terms which may or may not involve $\Delta x_1, \Delta y_1$

$$= 0.$$

This gives for the slope, $\dfrac{\Delta y_1}{\Delta x_1}$, an expression which involves $\Delta x_1, \Delta y_1$ in a different form; from this the slope of the tangent is found by making $\Delta x_1 = 0$, $\Delta y_1 = 0$.

Examples.

Apply this process to find the tangent at (x_1, y_1) to—

1. $3xy^2 = c.$
2. $2x^2 + 5xy + 2y^2 - 7x - 6y - 1 = 0.$
3. $y^2 = x(x - 1)(x - 2).$
4. $(x^2 + y^2)^2 = x^2 - y^2.$
5. Show that the equation of the tangent at (x_1, y_1) to—

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

is $x(ax_1 + hy_1 + g) + y(hx_1 + by_1 + f) + (gx_1 + fy_1 + c) = 0.$

94. When the tangent at a point (x_1, y_1) has been found, the normal can be written down by the process of § 36, for it is the line through (x_1, y_1) perpendicular to the tangent.

For example, the point $\left(4, \dfrac{12}{5}\right)$ is on $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$. The tangent at this point is—

$$\frac{4x}{25} + \frac{12}{5} \cdot \frac{y}{16} = 1,$$

$$16x + 15y = 100;$$

the normal is therefore—

$$15(x - 4) - 16\left(y - \frac{12}{5}\right) = 0,$$

that is,
$$15x - 16y - 60 + \frac{192}{5} = 0,$$

or
$$75x - 80y - 108 = 0.$$

95. If T, G are the points where the axis of x is met by the tangent and normal at a point P on any curve, PT is often spoken of as the length of the tangent, PG as the length of the normal. The part of the axis included between the foot of the ordinate and the point T or G, is the *subtangent*, or *subnormal.* In Fig. 59, PT and PG are

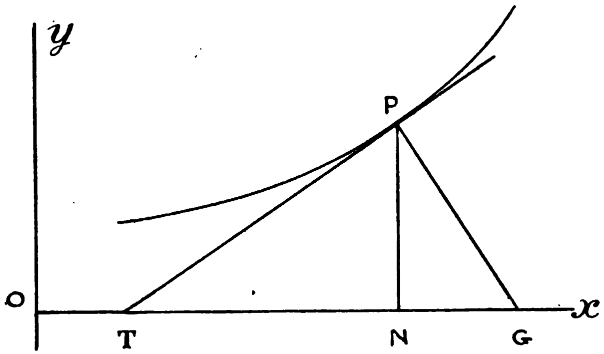


Fig. 59.

the tangent and normal, NT and NG are the subtangent and subnormal.

EXAMPLES.

1. Write down the equations of the tangent and normal to—

$$\frac{x^2}{25} + \frac{y^2}{16} = 1 \text{ at } \left(3, \frac{16}{5}\right).$$

2. Write down the tangent and normal to $x^2 + 4y^2 = 100$ at the points (8, 3), (− 6, 4).

3. Write down the tangent and normal to $y^2 = 3x$ at (12, 6).

4. Find the normal to $y = x^3$ at the points (1, 1), (2, 8).

5. Find the normal to $y^2 = x^3$ at (4, 8).

6. Write down the equation of the normal to $x^2 + y^2 = r^2$ at (x_1, y_1). From the equation prove that the normal at any point on a circle passes through the centre.

7. Find the tangent and normal at (x_1, y_1) to the circle—

$$(x - p)^2 + (y - q)^2 = r^2.$$

8. Find the normal to $y^2 = 4px$ at (x_1, y_1).

9. Find the normal to $\dfrac{x^2}{\alpha} + \dfrac{y^2}{\beta} = 1$ at (x_1, y_1).

10. Prove, for the parabola $y^2 = 4px$, SG = SP = ST.

11. Prove for the parabola that the subnormal is constant, $= 2p$, and that the subtangent is bisected at the vertex.

12. Prove, for the central conics, SG = e . SP.

13. Prove, for the central conics,

$$S'G : GS = S'P : PS = - S'T : TS.$$

96. In Chapter VI two methods were used for finding the tangents from a point; one made use of the line-equation (§ 89), the other determined the slopes of the tangents (§ 82). An alternative process leads to the determination of the points of contact, from which the tangents themselves are known.

Example.—Find tangents to $x^2 + 4y^2 = 100$ from the point (2, 7). Let (x_1, y_1) be the point of contact of a tangent from (2, 7).

The tangent at (x_1, y_1) is $\quad xx_1 + 4yy_1 = 100.$

This passes through (2, 7) if $\quad 2x_1 + 28y_1 = 100,$

that is, if $\quad x_1 + 14y_1 = 50.$

Thus the values x_1, y_1, which we already know must satisfy the equation $x^2 + 4y^2 = 100$, we now learn must satisfy also $x + 14y = 50$.

These equations give for y the quadratic—

$$\begin{aligned}(14y - 50)^2 + 4y^2 &= 100,\\ (7y - 25)^2 + y^2 &= 25,\\ 50y^2 - 350y + 600 &= 0,\\ y^2 - 7y + 12 &= 0,\\ (y - 3)(y - 4) &= 0.\end{aligned}$$

If $y = 3$, $x = 8$; if $y = 4$, $x = -6$.

Hence the solutions of the two equations are $(8, 3)$, $(-6, 4)$, and the given conditions are satisfied by taking for the point (x_1, y_1) either of the two points $(8, 3)$, $(-6, 4)$. The tangent at $(8, 3)$ is—

$$8x + 12y = 100,$$

that is,

$$2x + 3y = 25;$$

and the tangent at $(-6, 4)$ is—

$$-6x + 16y = 100,$$

that is,

$$-3x + 8y = 50.$$

97. In this example the coordinates of the point of contact (x_1, y_1) are determined by the condition that they must satisfy, not only the equation $x^2 + 4y^2 = 100$, which expresses that the desired point lies on the given ellipse, but also the equation $x + 14y = 50$. The latter is the equation of a straight line; the points of contact of tangents from $(2, 7)$ are determined as the intersections of the ellipse with this straight line. It will now be proved that the points of contact of tangents from any point (x', y') to a conic are obtained by means of a certain straight line, whose equation can be written down at once.

(i) *Parabola.*—The tangent to $y^2 = 4px$ at a point (x_1, y_1) is $yy_1 = 2p(x + x_1)$. This passes through the given point (x', y') if $y'y_1 = 2p(x_1 + x')$; the coordinates x_1, y_1 are to be found from this equation, combined with the equation that expresses that (x_1, y_1) lies on the parabola.

This may be stated in the form— x_1, y_1 are any values that satisfy $y^2 = 4px$ and $yy' = 2p(x + x')$. The geometrical interpretation of this is that the desired point is any point common to the parabola *and the straight line* $yy' = 2p(x + x')$.

(ii) *Central conics.*—The tangent to $\frac{x^2}{\alpha} + \frac{y^2}{\beta} = 1$ at a point (x_1, y_1) is $\frac{xx_1}{\alpha} + \frac{yy_1}{\beta} = 1$. This passes through the given point (x', y') if $\frac{x'x_1}{\alpha} + \frac{y'y_1}{\beta} = 1$. The geometrical interpretation of this equation of the first degree in x_1, y_1 is that (x_1, y_1) is some point on the locus $\frac{x'x}{\alpha} + \frac{y'y}{\beta} = 1$; hence (x_1, y_1) is any point common to the conic *and the straight line* $\frac{xx'}{\alpha} + \frac{yy'}{\beta} = 1$.

Since the line meets the conic (parabola, ellipse, or hyperbola) in two points, either of which may be taken as the point of contact of a tangent that shall go through (x', y'), there are two tangents; the line found is the chord through the two points of contact, the "chord of contact." The line is called the *polar* of the point (x', y') with respect to the conic; the point is called the *pole* of the line.

The polar of (x', y') with respect to the parabola $y^2 = 4px$ is $yy' = 2p(x + x')$; with respect to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ it is $\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1$; with respect to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ it is $\frac{xx'}{a^2} - \frac{yy'}{b^2} = 1$.

98. The polar of a real point is a real line, whether real tangents can be drawn from the point or not. For example, the polar of (1, 1) with respect to $x^2 + 4y^2 = 100$ is $x + 4y = 100$; this meets the given conic where

$$y^2 - 40y + 495 = 0,$$

and therefore at imaginary points.

Note.—Since $x^2 + 4y^2 - 100$ has a negative value at (1, 1), it is clear from the first that the tangents from (1, 1) are imaginary.

The polar of a focus of a conic is the corresponding

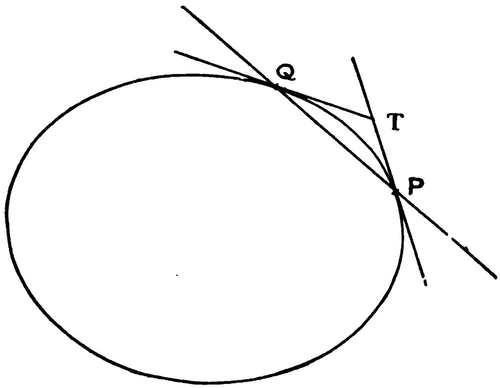


Fig. 60.

directrix. For the focus of $y^2 = 4px$ is $(p, 0)$, the polar of this, by the formula $yy' = 2p(x + x')$, is $0 = 2p(x + p)$, that is, $x + p = 0$, which is the directrix; and the focus of $\frac{x^2}{\alpha} + \frac{y^2}{\beta} = 1$ is $(ae, 0)$, whose polar is $\frac{x \cdot ae}{\alpha} + \frac{y \cdot 0}{\beta} = 1$, that is, $x = \frac{\alpha}{ae} = \frac{a^2}{ae} = \frac{a}{e}$, which is the directrix.

If (x', y') is a point on the curve, the equation found for the polar of (x', y') shows that this is the same line as the tangent at (x', y'). The reason for this may be seen with the help of Fig. 60; in this TP, TQ are tangents from T, PQ is the polar of T; as T approaches the curve, Q moves up to P. Ultimately the points P, Q are indistinguishable, and the chord PQ becomes a tangent.

99. If the line is given, the pole can be found. One way is to find the coordinates of the points of intersection of the line with the curve, then to write down the equations of the tangents at these points, and to find the coordinates of the point common to these two tangents by solving the equations. But there is a better way, illustrated in the following example.

Example.—Find the pole of each of the lines $3x - y + 8 = 0$, $y = 2x$ with respect to the curves $y^2 = 12x$, $x^2 + 4y^2 = 12$.

(i) $y^2 = 12x$.

If the pole is (x', y'), the polar is $yy' = 6(x + x')$. Hence x', y' must be determined so that this is the same as the given equation. The line $y = \frac{6}{y'}(x + x')$ is the same as $y = 3x + 8$ if $\frac{6}{y'} = 3$, $\frac{6x'}{y'} = 8$. Hence the pole is $\left(\frac{8}{3}, 2\right)$. It is the same as $y = 2x$ if $\frac{6}{y'} = 2$, $\frac{6x'}{y'} = 0$. Hence the pole is (0, 3).

(ii) $x^2 + 4y^2 = 12$.

The polar of (x', y') is $xx' + 4yy' = 12$, that is, $y = -\frac{x'}{4y'}x + \frac{3}{y'}$; this is the same as $y = 3x + 8$ if $-\frac{x'}{4y'} = 3, \frac{3}{y'} = 8$. Hence the pole is $\left(-\frac{9}{2}, \frac{3}{8}\right)$.

To make it the same as $y = 2x$, we must have $-\frac{x'}{4y'} = 2$, $\frac{3}{y'} = 0$.

These equations give $y' = \infty$, $x' = \infty$. Hence the pole is at infinity, but in a definite direction, since $y' = -\frac{1}{8}x'$. The pole of $y = 2x$ is at infinity on the line $y = -\frac{1}{8}x$.

EXAMPLES.

1. Find the polar of $\left(-\frac{2}{3}, \frac{5}{3}\right)$ with respect to $y^2 = 4x$. Apply this to find the tangents from this point.

2. Find the polar of (15, 7) with respect to $\frac{x^2}{25} + \frac{y^2}{16} = 1$. Apply this to find the tangents from this point.

3. Find the poles of the lines $2x + 3y + 1 = 0$, $x = 4$, $y = 1$, $x + y = 0$ with respect to $y^2 = 16x$.

4. Find the poles of these lines with respect to $x^2 + y^2 = 50$, $\frac{x^2}{9} + \frac{y^2}{8} = 1$, $\frac{x^2}{6} - \frac{y^2}{8} = 1$.

5. Show that the polar of (x', y') with respect to a circle is perpendicular to the line that joins (x', y') to the centre of the circle.

100. From the equation found for the polar of a point U, (x', y'), it follows that if V, (x'', y''), is any point on this polar, the polar of V passes through U. For the polar of (x', y'), is, for the

parabola $y^2 = 4px$ $yy' = 2p(x + x')$.	*central conic* $\frac{x^2}{\alpha} + \frac{y^2}{\beta} = 1$, $\frac{xx'}{\alpha} + \frac{yy'}{\beta} = 1$.

The condition that these pass through (x'', y'') is—

$y''y' = 2p(x'' + x')$,	$\frac{x''x'}{\alpha} + \frac{y''y'}{\beta} = 1$.

Similarly the polar of (x'', y'') is—

$yy'' = 2p(x + x'')$,	$\frac{xx''}{\alpha} + \frac{yy''}{\beta} = 1$,

and the condition that these pass through (x', y') is—

$$y'y'' = 2p(x' + x''), \quad \Big| \quad \frac{x'x''}{\alpha} + \frac{y'y''}{\beta} = 1,$$

the same condition as before. That is, if the polar of U passes through V, the polar of V passes through U. The points U, V are said to be *conjugate;* thus a point is conjugate to every point on its polar.

This result can be formulated with reference to the lines instead of the points. Denote the polars of points by the corresponding small letters; the above statement becomes—if u passes through V, then v passes through U. This is equivalent to, if U lies on v, then V lies on u; that is, if the pole of u lies on v, then the pole of v lies on u. The lines u, v are said to be conjugate; thus a line is conjugate to every line through its pole. It is the same thing to say that two points are conjugate as that their polars are conjugate.

101. The true significance of the properties of poles and polars is made clearer by their expression in terms of the harmonic relation (§ 9). If a chord PQ of a conic is divided by the points U, V internally and externally in the same ratio, then U, V, which are harmonic conjugates with respect to P, Q, are said to be conjugate with respect to the conic; and in this case the polar of either point passes through the other. For if PQ is divided at U in the ratio $k:1$, hence at V in the ratio $-k:1$, the coordinates of U are $x' = \dfrac{x_1 + kx_2}{1+k}$, $y' = \dfrac{y_1 + ky_2}{1+k}$, and the

coordinates of V are $x'' = \dfrac{x_1 - kx_2}{1 - k}$, $y'' = \dfrac{y_1 - ky_2}{1 - k}$, where the coordinates of P, Q are (x_1, y_1), (x_2, y_2).

The polar of U is
$$\frac{xx'}{\alpha} + \frac{yy'}{\beta} = 1,$$

that is,
$$\frac{x}{\alpha} \cdot \frac{x_1 + kx_2}{1 + k} + \frac{y}{\beta} \cdot \frac{y_1 + ky_2}{1 + k} = 1;$$

this passes through V if
$$\frac{x''}{\alpha} \cdot \frac{x_1 + kx_2}{1 + k} + \frac{y''}{\beta} \cdot \frac{y_1 + ky_2}{1 + k} = 1,$$

that is, if
$$\frac{1}{\alpha} \cdot \frac{x_1 - kx_2}{1 + k} \cdot \frac{x_1 + kx_2}{1 + k} + \frac{1}{\beta} \frac{y_1 - ky_2}{1 - k} \cdot \frac{y_1 + ky_2}{1 + k} = 1.$$

The condition is therefore—
$$\frac{1}{\alpha} \cdot \frac{x^2 - k^2x_2^2}{1 - k^2} + \frac{1}{\beta} \cdot \frac{y_1^2 - k^2y_2^2}{1 - k^2} = 1,$$

that is,
$$\frac{1}{\alpha}(x_1^2 - k^2x_2^2) + \frac{1}{\beta}(y_1^2 - k^2y_2^2) = 1 - k^2,$$

or
$$\left(\frac{x_1^2}{\alpha} + \frac{y_1^2}{\beta} - 1\right) - k^2\left(\frac{x_2^2}{\alpha} + \frac{y_2^2}{\beta} - 1\right) = 0,$$

which is satisfied, since both points (x_1, y_1), (x_2, y_2) lie on the conic.

Thus when we speak of points as conjugate, because the polar of each passes through the other, we mean that they are harmonic conjugates with respect to the conic.

EXAMPLE.

Apply this method to prove the theorem for the parabola.

102. The theorem that if V be any point on u, the polar of V passes through u, shows that the polars of all points on a line u pass through a point U; that is, collinear poles have concurrent polars, and concurrent polars have collinear poles.

Let the lines u, v meet at W, then w, the polar of W, passes through both U and V (Fig. 61). If u, v are conjugate, so that U is on v and V on u, the three lines

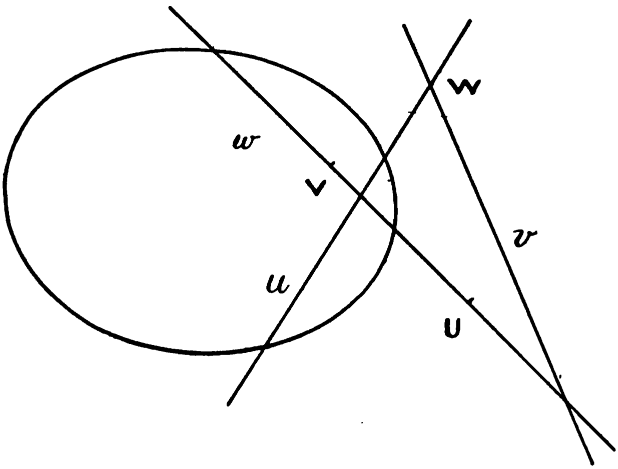


Fig. 61.

u, v, w are the sides of a triangle of which U, V, W are the vertices (Fig. 62). The triangle UVW, which has the property that each vertex is the pole of the opposite side,

each side the polar of the opposite vertex, is called a self-polar or self-conjugate triangle, or a harmonic triangle, with respect to the conic.

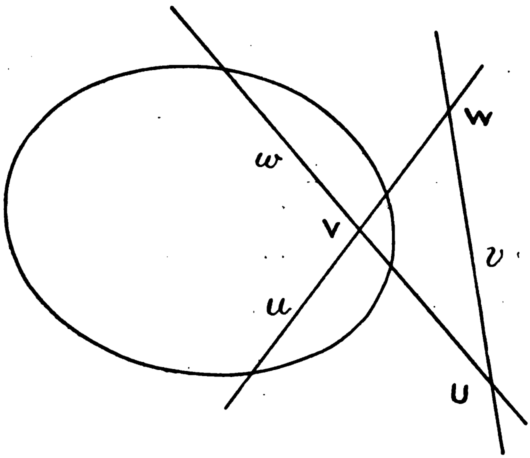


Fig. 62.

103. The condition that two points (or two lines) be conjugate is found by expressing that one lies on the polar (passes through the pole) of the other.

(i) *Parabola.*—If the coordinates of the pole are x', y', the equation of the polar is—

$$yy' = 2p(x + x'),$$

that is,
$$\frac{1}{x'}.x - \frac{y'}{2px'}.y + 1 = 0;$$

hence the coordinates of the polar are—

$$\xi' = \frac{1}{x'}, \quad \eta' = -\frac{y'}{2px'}.$$

These give also the coordinates of the pole in terms of the coordinates of the polar—

$$x' = \frac{1}{\xi'},\ y' = \frac{-2p\eta'}{\xi'}.$$

The condition that two points (x', y') (x'', y'') be conjugate is $y'y'' = 2p(x' + x'')$; and the condition that two lines (ξ', η'), (ξ'', η'') be conjugate is—

$$\frac{-2p\eta'}{\xi'} \times \frac{-2p\eta''}{\xi''} = 2p\left(\frac{1}{\xi'} + \frac{1}{\xi''}\right),$$

that is, $$\eta'\eta'' = \frac{1}{2p}(\xi' + \xi'').$$

(ii) *Central conics.*—The polar of (x', y') is—

$$\frac{xx'}{\alpha} + \frac{yy'}{\beta} = 1\ ;$$

hence the coordinates of the polar are—

$$\xi' = -\frac{x'}{\alpha},\ \eta' = -\frac{y'}{\beta},$$

from which $$x' = -\alpha\xi',\ y' = -\beta\eta'.$$

The condition that two points (x', y'), (x'', y'') be conjugate is—

$$\frac{x'x''}{\alpha} + \frac{y'y''}{\beta} = 1\ ;$$

and the condition that two lines (ξ', η'), (ξ'', η'') be conjugate is $\dfrac{-\alpha\xi' \times -\alpha\xi''}{\alpha} + \dfrac{-\beta\eta' \times -\beta\eta''}{\beta} = 1$,

that is, $$\alpha\xi'\xi'' + \beta\eta'\eta'' = 1.$$

Note.—The condition that two lines be conjugate is related to the line-equation of the conic precisely as the condition that two points be conjugate is related to the point-equation.

Example i.—Find the line through the point (7, 2) conjugate to $x + y = 25$ with respect to $x^2 + 4y^2 = 100$.

First method.—The desired line is to be conjugate to $x + y = 25$, hence it must pass through the pole of this line. Let this pole be (x', y'); then $xx' + 4yy' = 100$ is the same as $x + y = 25$. Hence $x' = 4$, $y' = 1$. The line sought is to pass through the two points (4, 1), (7, 2); its equation is therefore $\frac{x-4}{7-4} = \frac{y-1}{2-1}$, that is—

$$x - 3y - 1 = 0.$$

Second method.—The coordinates of the given line are $-\frac{1}{25}, -\frac{1}{25}$; let the desired line be (ξ', η'). The condition that these lines be conjugate with respect to $\frac{x^2}{100} + \frac{y^2}{25} = 1$, that is, $100\xi^2 + 25\eta^2 = 1$, is $100 \times -\frac{1}{25} \times \xi' + 25 \times -\frac{1}{25} \times \eta' = 1$, that is, $4\xi' + \eta' + 1 = 0$.
The condition that the line (ξ', η') may pass through (7, 2) is—

$$7\xi' + 2\eta' + 1 = 0.$$

Hence $\xi' = -1$, $\eta' = 3$; the line is $-x + 3y + 1 = 0$, as before.

Example ii.—Find a line perpendicular to $x + y = 25$, and also conjugate to this same line with respect to $x^2 + 4y^2 = 100$.

If the line be (ξ', η'), it is perpendicular to the given line if $\xi' + \eta' = 0$, and conjugate if $4\xi' + \eta' + 1 = 0$. Hence—

$$\xi' = -\frac{1}{3}, \eta' = \frac{1}{3};$$

the line is $$-\frac{1}{3}x + \frac{1}{3}y + 1 = 0,$$

that is, $$x - y - 3 = 0.$$

Or, by the other method, the line is to pass through the point (4, 1), and to be perpendicular to $x + y = 25$, hence its equation is $(x - 4) - (y - 1) = 0$, that is, $x - y - 3 = 0$.

Example iii.—Prove that any line through a focus of a conic is perpendicular to its conjugate through that focus.

(Proof for central conics.)

The condition that a line (ξ', η') pass through the focus $(ae, 0)$ is—

$$ae\xi' + 1 = 0. \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The line (ξ, η) and its conjugate (ξ', η') satisfy—

$$a\xi\xi' - b\eta\eta' = 1 \quad . \quad . \quad . \quad . \quad (2)$$

Since the conjugate line is to be the one that passes through the focus,

$$ae\xi' - 1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From equations (1), (2), (3) it is to be proved that—

$$\xi\xi' - \eta\eta' = 0.$$

From (1) and (3), $\quad$ [illegible] $\xi\xi' = 1$[illegible]

that is, $\quad$ $a - b\xi\xi' = 1$[illegible]

This subtracted from (2) gives—

$$b\xi\xi' - \eta\eta' = 0,$$

that is, $\quad$ $\xi\xi' - \eta\eta' = 0,$

which proves that the lines are perpendicular.

Examples.

1. Find a point conjugate to both points (1, 2), (10, 6) with respect to

$$\frac{x^2}{25} + \frac{y^2}{16} = 1.$$

2. Also with respect to $y^2 = 12x$.

3. Also with respect to $xy = 20$.

4. Find a point conjugate to (5, 4) with respect to both curves,

$$\frac{x^2}{25} + \frac{y^2}{16} = 1,\ 15y^2 + 32x = 0.$$

5. Find a line conjugate to both lines,

$$x + y - 6 = 0,\ 7x + y + 1 = 0,$$

with respect to the curves,

$$\text{(i)}\ \frac{x^2}{25} + \frac{y^2}{16} = 1,\ \text{(ii)}\ y^2 = 12x,\ \text{(iii)}\ xy = 20.$$

6. Show that the points (4, 1), (36, − 11) are conjugate with respect to $x^2 + 4y^2 = 100$. Find the third vertex of the self-conjugate triangle of which these two points are vertices.

7. Prove that any line through the focus of a parabola is perpendicular to its conjugate through the focus.

8. Two points are conjugate with respect to $x^2 + 4y^2 = 100$, and also with respect to $2x^2 - y^2 = 92$. Show that they are conjugate with respect to $x^2 + y^2 = 64$.

9. Two points are conjugate with respect to each of the curves,

$$fx^2 + gy^2 = 1, f'x^2 + g'y^2 = 1.$$

Show that they are conjugate with respect to

$$(f + kf')x^2 + (g + kg')y^2 = 1 + k$$

for all values of k.

10. Find the equation of the tangent at a point (x_1, y_1) on

$$ax^2 + by^2 + 2gx + c = 0.$$

11. Find the polar of a point with respect to

$$ax^2 + by^2 + 2gx + c = 0.$$

12. Show that the two conics

$$ax^2 + by^2 + 2gx + c = 0,$$
$$ax^2 + by^2 + 2g'x + c = 0,$$

have one common chord along the axis of y.

13. Two points are conjugate with respect to each of the conics,

$$ax^2 + by^2 + 2gx + c = 0.$$

Prove (i) that they are at equal distances from the common chord, but on opposite sides; (ii) that they are conjugate with respect to $ax^2 + by^2 + 2kx + c = 0$ for all values of k.

14. Find the intercepts made on the axes by the polar of (x', y') with respect to $\dfrac{x^2}{a} + \dfrac{y^2}{\beta} = 1$.

15. Find the locus of a point, if the part of its polar with respect to $\dfrac{x^2}{a} + \dfrac{y^2}{\beta} = 1$ that is included between the axes is of constant length. Find also the envelope of the polar.

16. Find the locus of a point, if its polar with respect to $\dfrac{x^2}{a} + \dfrac{y^2}{\beta} = 1$, and the axes, include a triangle of constant area.

17. Find the locus of a point which moves so as to remain at a constant distance from its polar with respect to $\frac{x^2}{\alpha} + \frac{y^2}{\beta} = 1$. Find also the envelope of the polar.

18. A line moves, remaining at a constant distance k from the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Find the (point-equation of the) locus of the pole with respect to this ellipse.

19. A point moves, remaining at a constant distance k from the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Find the (line-equation of the) envelope of the polar with respect to this ellipse.

20. Show that a point conjugate to itself with respect to any conic is a point on the conic, and that a line conjugate to itself is a tangent to the conic.

104. When a point (x', y') moves in any manner, its polar (ξ', η') moves also. Since ξ', η' are known in terms of x', y', and x', y' in terms of ξ', η', if the condition to which the coordinates of the point are subject is known, the condition to which the coordinates of the line are subject can be found, and *vice-versa*. That is, if the locus of a point is given, the envelope of its polar with respect to a given conic can be found; and if the envelope of a line is given, the locus of the pole can be found. For instance, Examples 18, 19, above, may be stated in the form—

A line envelopes $x^2 + y^2 = k^2$; find the locus of its pole with respect to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

A point describes $x^2 + y^2 = k^2$, find the envelope of its polar with respect to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

For a reason that will be explained in Chapter XII., the two curves which are respectively described by the pole and enveloped by the polar are called *reciprocal polars* with respect to the given (fundamental) conic.

Example.—A point describes the curve

$$2x^2 + 5xy + 2y^2 + 4x + 2y - 1 = 0;$$

find the envelope of its polar with respect to $x^2 + 4y^2 = 100$.

The polar of (x', y') is $xx' + 4yy' = 100$; the coordinates (ξ', η') of the polar of (x', y') are therefore—

$$\xi' = -\frac{x'}{100}, \ \eta' = -\frac{y'}{25}.$$

The problem is now, x', y' satisfy the equation—

$$2x'^2 + 5x'y' + 2y'^2 + 4x' + 2y' - 1 = 0;$$

to what relation are ξ', η' subject?

Since $x' = -100\xi'$, $y' = -25\eta'$, the desired result is obtained by direct substitution; it is—

$$2 \,.\, 100^2\xi'^2 + 5 \,.\, 100 \,.\, 25\xi'\eta' + 2 \,.\, 25^2 \,.\, \eta'^2 - 400\xi' - 50\eta' - 1 = 0;$$

the envelope of the polar is therefore—

$$20000\xi^2 + 12500\xi\eta + 1250\eta^2 - 400\xi - 50\eta - 1 = 0.$$

In general, if the point describes a given curve, its coordinates satisfy an equation of the form $f(x', y') = 0$; by direct substitution of the expressions for x', y' in terms of ξ', η' this becomes $F(\xi', \eta')$, which is the equation of the envelope of the polar; and if the line envelopes a curve, its coordinates satisfy an equation $f(\xi', \eta')$, which, by substitution of the expressions for ξ', η' in terms of x', y', becomes $F(x', y')$, the equation of the locus of the pole.

105. The problem may be solved by other methods.

Example.—A line envelopes $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Find the locus of its pole with respect to $x^2 + y^2 = r^2$.

Any tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $y = mx \pm \sqrt{a^2m^2 + b^2}$.

If (x', y') is the pole of $y = mx + \sqrt{a^2m^2 + b^2}$ with respect to $x^2 + y^2 = r^2$, this equation is the same as $xx' + yy' = r^2$, that is, as

$$y = -\frac{x'}{y'}x + \frac{r^2}{y'}.$$

Hence
$$m = -\frac{x'}{y'}, \quad \sqrt{a^2m^2 + b^2} = \frac{r^2}{y'}.$$

The elimination of m from these two equations will give a relation connecting x', y'. This is—

$$\sqrt{a^2\frac{x'^2}{y'^2} + b^2} = \frac{r^2}{y'},$$

that is,
$$\sqrt{a^2x'^2 + b^2y'^2} = r^2.$$

The locus of the point (x', y') is therefore the curve,

$$a^2x^2 + b^2y^2 = r^4.$$

Or, by another method. The tangent at (x_1, y_1) is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$. If the pole with respect to $x^2 + y^2 = r^2$ is (x', y'), this equation is the same as $xx' + yy' = r^2$.

Hence
$$x' = \frac{r^2x_1}{a^2}, \quad y' = \frac{r^2y_1}{b^2};$$

that is,
$$x_1 = \frac{a^2x'}{r^2}, \quad y_1 = \frac{b^2y'}{r^2}.$$

The problem can now be stated in the form, find the relation to which x', y' are subject when x_1, y_1 satisfy $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$. Direct substitution of the values for x_1, y_1 in terms of x', y' gives the result—

$$\frac{a^4x'^2}{r^4a^2} + \frac{b^4y'^2}{r^4b^2} = 1,$$

that is,
$$a^2x'^2 + b^2y'^2 = r^2.$$

Examples.

1. A line envelopes $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=k^2$. Show that the locus of its pole with respect to $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=\dfrac{1}{k^2}$.

2. A line envelopes $\dfrac{k^2x^2}{a^4}+\dfrac{k^2y^2}{b^4}=1$; find the locus of its pole with respect to $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. Interpret the result.

3. A point describes $\dfrac{k^2x^2}{a^4}+\dfrac{k^2y^2}{b^4}=1$; find the envelope of its polar with respect to $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. Interpret the result.

4. A line envelopes $y^2=4px$. Show that the locus of its pole with respect to $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is of the form $y^2=4qx$. Find the relation satisfied by p and q.

5. A point P describes the circle $x^2+y^2-4ry+3r^2=0$; a line through P parallel to the axis of y meets the polar of P, with respect to $x^2+y^2=r^2$, at Q. Find the locus of Q.

6. Two points are conjugate with respect to $x^2+y^2=r^2$, and lie on a straight line through the origin. Express the coordinates of one of the two points in terms of those of the other. Show that if one of the points describes a straight line, the other describes a circle through the origin.

7. Show that inverse points, defined on p. 91, are conjugate with respect to the circle.

8. Two lines, conjugate with respect to $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, pass each through one focus. Show that the locus of their intersection is an ellipse through the foci.

106. The normals from a point (h, k) are determined by the process used in §§ 96, 97 for finding the tangents from a point. Write down the equation of the normal at a point (x_1, y_1) on the curve, express that this passes through the given point (h, k); this gives an equation in x_1, y_1,

which, combined with the equation expressing that (x_1, y_1) is a point on the given curve, determines a certain number of values for x_1, y_1.

(i) *Parabola.*—The tangent at (x_1, y_1) to $y^2 = 4px$ is—

$$yy_1 = 2p(x + x_1),$$

the normal is therefore—

$$y_1(x - x_1) + 2p(y - y_1) = 0.$$

This passes through (h, k) if

$$y_1(h - x_1) + 2p(k - y_1) = 0.$$

Hence the points at which the normals must be taken, in order that they may pass through (h, k), lie on the curve

$$xy - (h - 2p)y - 2pk = 0.$$

The points are given by this equation, combined with $y^2 = 4px$, hence by the equation in y,

$$y^3 - 4p(h - 2p)y - 8p^2k = 0.$$

Since this is a cubic equation, we learn that three normals to a parabola pass through a point (h, k).

(ii). *Central conics.*—The tangent at (x_1, y_1) to

$$\frac{x^2}{\alpha} + \frac{y^2}{\beta} = 1$$

is

$$\frac{xx_1}{\alpha} + \frac{yy_1}{\beta} = 1\,;$$

the normal is therefore—

$$\frac{\alpha}{x_1}(x - x_1) - \frac{\beta}{y_1}(y - y_1) = 0.$$

This passes through (h, k) if

$$\frac{\alpha}{x_1}(h - x_1) - \frac{\beta}{y_1}(k - y_1) = 0,$$

that is, if $\quad \alpha y_1(h - x_1) - \beta x_1(k - y_1) = 0,$

$$(\alpha - \beta)x_1 y_1 + \beta k x_1 - \alpha h y_1 = 0.$$

Hence the points at which the normals must be taken, in order that they may pass through (h, k), lie on the curve,

$$(\alpha - \beta)xy + \beta kx - \alpha hy = 0.$$

The points are given by this equation, combined with $\frac{x^2}{\alpha} + \frac{y^2}{\beta} = 1$. Elimination of y gives for x an equation of the fourth degree, hence four normals to an ellipse or hyperbola pass through a point (h, k).

EXAMPLES.

1. The normals to a parabola at the three points P, Q, R meet in a point. Show that the sum of the ordinates of P, Q, R is zero.

2. From the point of intersection of the normals to $y^2 = 4x$ at $(1, 2)$, $\left(\frac{9}{4}, 3\right)$ the third normal is drawn. Find the coordinates of the point on the parabola at which this line is the normal.

3. Find the envelope of normals to a parabola.

4. Find the envelope of normals to a central conic.

5. Find the locus of the foot of the perpendicular from the vertex of a parabola to a normal.

6. Find the locus of the foot of the perpendicular from the focus of a parabola to a normal.

7. Find the locus of the foot of the perpendicular from the centre of a conic to a normal.

8. Find the locus of the foot of the perpendicular from a focus of a central conic to a normal.[1]

9. Show that the locus of the point of intersection of two normals to a parabola at the extremities of a chord whose direction is fixed is a normal to the parabola.

10. Ellipses are described with the same major axis, but different minor axes. Prove that tangents at points on a line perpendicular to the major axis meet on the major axis.

11. The extremities of chords of the ellipses in Ex. 10 lie on two lines perpendicular to the major axis; prove that the chords meet on the major axis.

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[1] Examples 7 and 8. Solve the equation of the normal at (x_1, y_1), and the equation of the perpendicular, for $\frac{\alpha}{x_1}, \frac{\beta}{y_1}$; hence obtain x_1, y_1; express that these satisfy $\frac{x^2}{\alpha}+\frac{y^2}{\beta} = 1$.

CHAPTER VIII

BISECTED CHORDS. DIAMETERS

107. THIS chapter contains theorems relating to the bisections of chords of a conic. To begin with, one of the methods employed is illustrated by numerical examples.

Example i.—Find the point of bisection of the chord of $y^2 = 4x$ that lies on the line $y = 5x - 7$.

The equation for intersections is—

$$(5x - 7)^2 = 4x,$$

that is,
$$25x^2 - 74x + 49 = 0.$$

Instead of solving this equation for x_1, x_2, the abscissæ of the common points, and then finding the abscissa of the point of bisection, x', from the relation $x' = \frac{x_1 + x_2}{2}$, write down at once $x_1 + x_2 = \frac{74}{25}$ (by the formula for the sum of the roots of a quadratic). Hence for the point of bisection, $x' = \frac{37}{25}$; and as it lies on the given line, $y' = 5x' - 7 = \frac{2}{5}$. The point of bisection is $\left(\frac{37}{25}, \frac{2}{5}\right)$.

Example ii.—Find the point of bisection of the chord of $\frac{x^2}{100} + \frac{y^2}{25} = 1$ that lies on $x - 7y + 6 = 0$.

The equation for intersections is—

$$\frac{(7y - 6)^2}{100} + \frac{y^2}{25} = 1,$$

that is,
$$53y^2 - 84y - 64 = 0.$$

Hence $y_1 + y_2 = \frac{84}{53}$, and for the point of bisection,

$$y' = \frac{y_1 + y_2}{2} = \frac{42}{53},$$

$$x' = 7y' - 6 = -\frac{24}{53}.$$

The point is $\left(-\frac{24}{53}, \frac{42}{53}\right)$.

Example iii.—Find the locus of the points of bisection of chords of $x^2 + 4y^2 = 100$ with slope 2.

Any such line is $y = 2x + n$. The equation for intersections is—

$$x^2 + 4(2x + n)^2 = 100,$$

that is, $$17x^2 + 16nx + 4n^2 - 100 = 0.$$

Hence $$x' = \frac{x_1 + x_2}{2} = -\frac{8n}{17},$$

$$y' = 2x' + n = \frac{n}{17}.$$

To find the locus of (x', y') for all values of n, eliminate n; this gives $\frac{y'}{x'} = -\frac{1}{8}$, hence the locus of the points of bisection is the straight line $x + 8y = 0$.

Examples.

1. Find the points of bisection of the chords of $y^2 = 4x$ that lie on $y = 5x - 1$, $y = 5x - 100$, $y = 5x + n$. Illustrate by a careful diagram.

2. Find the points of bisection of chords of $x^2 - y^2 = 1$ that lie on $3x - y + 5 = 0$, $3x - y + 1 = 0$, $3x - y = 0$, $3x - y + 9 = 0$. Illustrate by a careful diagram.

3. Find the locus of the points of bisections of chords with slope 5 for the curves $x^2 - y^2 = 1$, $9x^2 - y^2 = 36$, $xy = 4$.

4. Show that points of bisection of chords of $y^2 = 12x$ with slope 4 lie on the line $y = \frac{3}{2}$.

108. In these examples it is seen that the points of bisection of the particular system of parallel chords of the conic lie on a straight line, which is parallel to the axis if

the conic is a parabola, but passes through the centre if the conic is an ellipse or hyperbola. This theorem is now to be proved.

There are three methods of proving it—

1. Start with a chord with the given slope; form the equation for intersections, hence obtain the coordinates of he point of bisection.

2. Start with the point of bisection; write down the coordinates of two points at equal distances from this, lying in the given direction; express that these two points are on the conic.

3. Start with two points on the conic, from which the coordinates of the point of bisection can be written down; express that the chord joining these points has the given slope.

(i) *The central conics* $\frac{x^2}{\alpha} + \frac{y^2}{\beta} = 1$.

First method.—Any line with the given slope m is—

$$y = mx + n;$$

the different values of n give all lines with this slope.

The equation for intersections is—

$$\beta x^2 + \alpha(mx + n)^2 = \alpha\beta,$$

that is, $(\alpha m^2 + \beta)x^2 + 2\alpha mnx + \alpha(n^2 - \beta) = 0$.

If the roots of this are x_1, x_2, then $x_1 + x_2 = \frac{-2\alpha mn}{\alpha m^2 + \beta}$. Hence if (x', y') be the point of bisection,

$$x' = \frac{x_1 + x_2}{2} = -\frac{\alpha mn}{\alpha m^2 + \beta},$$

$$y' = mx' + n = \frac{\beta n}{\alpha m^2 + \beta}.$$

These values show that the point of bisection is different for different values of n, that is, for different chords; to find the locus for *all* values of n, eliminate n. The result is—

$$\frac{y'}{x'} = -\frac{\beta}{am},$$

that is,
$$y' = -\frac{\beta}{am}x'.$$

Hence the locus of the point of bisection of a chord with slope m is the straight line,

$$y = -\frac{\beta}{am}x,$$

which passes through the centre. By properly choosing m, this can be made to be *any* straight line through the centre.

Second method.—If R is the point of bisection (x', y'),

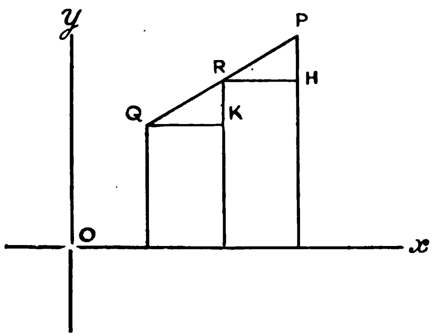

Fig. 63.

and P, Q two points at equal distances from R on the line with slope m (Fig. 63), then for the point P,

$$x_1 = x' + t,$$
$$y_1 = y' + mt,$$

where $$t = \text{RH},\ mt = \text{HP};$$

and for the point Q, since QR = RP and therefore QK = RH,

$$x_2 = x' - t,$$
$$y_2 = y' - mt.$$

Since P, Q are on the conic,

$$\beta(x' + t)^2 + \alpha(y' + mt)^2 = 1,$$
$$\beta(x' - t)^2 + \alpha(y' - mt)^2 = 1,$$

from which, by subtraction,

$$4\beta tx' + 4\alpha mty' = 0,$$

that is, $y' = -\dfrac{\beta}{\alpha m}x'$, as before.

Third method.—The chord that joins (x_1, y_1), (x_2, y_2) has the equation (§ 91)—

$$y - y_1 = -\frac{\beta(x_1 + x_2)}{\alpha(y_1 + y_2)}(x - x_1).$$

Hence $-\dfrac{\beta(x_1 + x_2)}{\alpha(y_1 + y_2)} = m$, the given slope.

Now $x' = \dfrac{x_1 + x_2}{2}$, $y' = \dfrac{y_1 + y_2}{2}$; hence this becomes $-\dfrac{2\beta x'}{2\alpha y} = m$, that is, $y' = -\dfrac{\beta}{\alpha m}x'$, as before.

(ii) *The parabola.*— $y^2 = 4px$.

First method.—The equation for intersections is—

$$(mx + n)^2 = 4px,$$

that is, $$m^2x^2 + 2(mn - 2p)x + n^2 = 0.$$

Hence
$$x' = \frac{x_1 + x_2}{2} = -\frac{mn - 2p}{m^2},$$
$$y' = mx' + n = \frac{2p}{m}.$$

No elimination is needed, for the value of y' does not depend on n; all chords with the slope m give for y' the same value. The locus of the point of bisection is therefore $y = \frac{2p}{m}$, which is a straight line parallel to the axis of the parabola. By properly choosing m, this can be made to be *any* straight line parallel to the axis.

Second method.—Write
$$x_1 = x' + t,$$
$$y_1 = y' + mt;$$
and therefore
$$x_2 = x' - t,$$
$$y_2 = y' - mt;$$
hence
$$(y' + mt)^2 = 4p(x' + t),$$
and
$$(y' - mt)^2 = 4p(x' - t),$$

from which, by subtraction,
$$4mty' = 8pt, \; y' = \frac{2p}{m}.$$

Third method.—The chord joining (x_1, y_1), (x_2, y_2) is (§ 91)—
$$y - y_1 = \frac{4p}{y_1 + y_2}(x - x_1).$$

Hence
$$\frac{4p}{y_1 + y_2} = m,$$
that is,
$$\frac{4p}{2y'} = m, \; y' = \frac{2p}{m}.$$

The theorem can be proved for any equation of the second degree, by any one of the three methods.

EXAMPLES.

1. Apply the first and second methods to show that chords of $2xy = k$ with slope m are bisected by the line $mx + y = 0$.

2. Apply the first and second methods to show that chords of $ax^2 + 2hxy + by^2 + c = 0$ with slope m are bisected by the line—

$$(ax + hy) + m(hx + by) = 0.$$

3. Apply the second method to show that chords of—

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

with slope m are bisected by the line—

$$(ax + hy + g) + m(hx + by + f) = 0.$$

109. The locus of the bisections of parallel chords of a conic is called a *diameter*. A diameter of a parabola is therefore any straight line parallel to the axis, and a diameter of an ellipse or hyperbola is any straight line through the centre.

If the diameter is given, the chords are known. For the parabola, if the diameter is $y = k$, the slope of the chords is known from the relation $\frac{2p}{m} = k$; hence $m = \frac{2p}{k}$. For the ellipse and hyperbola, if the diameter is $y = m'x$, then $m' = -\frac{\beta}{am}$; hence m, the slope of the chords bisected, $= -\frac{\beta}{am'}$.

110. The points where a diameter meets a conic are the

extremities of the diameter; the halves of the bisected chords are the ordinates with respect to the diameter. By the length of a diameter is meant the length of the part that is intercepted by the curve, when the diameter meets the curve in real points. (For a definition of the length of a diameter which does not meet the curve in real points, see Chap. IX., § 132.)

Parabola.—A diameter $y = \frac{2p}{m}$ of the parabola $y^2 = 4px$ meets the curve only at the point $\left(\frac{p}{m^2}, \frac{2p}{m}\right)$, at a finite distance; but it will be shown in Chapter IX. that the diameter meets the curve also at infinity. The tangent at this (finite) extremity is $y = mx + \frac{p}{m}$, which is parallel to the chords bisected. The polar of any point (x', y') on the diameter is—

$$yy' = 2p(x + x'),$$

that is, since $y' = \frac{2p}{m}$,

$$y = m(x + x');$$

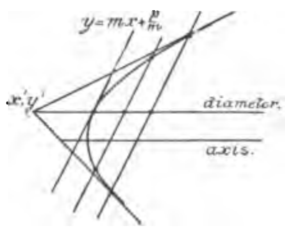

Fig. 64.

thus the polar belongs to the system of chords considered. Hence also the pole of any one of the bisected chords lies on the diameter, from which it follows that the tangents at the extremities of a chord meet on the diameter that bisects the chord (Fig. 64).

Central conics.—A diameter $y = -\frac{\beta}{am}x$

meets the conic $$\frac{x^2}{a} + \frac{y^2}{\beta} = 1$$

where $$\frac{x^2}{a^2m^2}(am^2 + \beta) = 1.$$

Hence $$x = \frac{+am}{\sqrt{am^2 + \beta}},\quad y = \frac{-\beta}{\sqrt{am^2 + \beta}},$$

or $$x = \frac{-am}{\sqrt{am^2 + \beta}},\quad y = \frac{+\beta}{\sqrt{am^2 + \beta}};$$

that is, the diameter has two extremities. The tangent at $\left(\frac{am}{\sqrt{am^2 + \beta}}, \frac{-\beta}{\sqrt{am^2 + \beta}}\right)$ is $y = mx - \sqrt{am^2 + \beta}$, and the tangent at $\left(\frac{-am}{\sqrt{am^2 + \beta}}, \frac{+\beta}{\sqrt{am^2 + \beta}}\right)$ is $y = mx + \sqrt{am^2 + \beta}$ (§ 79), hence the tangents at the extremities of a diameter are parallel to the chords bisected by the diameter.

The polar of any point (x', y') on the diameter is—

$$\frac{xx'}{a} + \frac{yy'}{\beta} = 1,$$

that is, since $$y' = -\frac{\beta}{am}x',$$

$$\frac{xx'}{a} - \frac{x'}{am} \cdot y = 1,$$

or $$y = m\left(x - \frac{a}{x'}\right),$$

which has the slope m. That is, the polar of any point on the diameter belongs to the system of chords considered. Thus also the pole of any one of the bisected chords lies

on the diameter, from which it follows that the tangents at the extremities of a chord meet on the diameter that bisects the chord.

111. So far the theory of diameters is the same for the parabola and the central conics. The theorems just given, which are true for both, are an immediate geometrical consequence of the fact that the diameter bisects parallel chords. For if the parallel chords PQ, P′Q′ are bisected at R, R′, we know from elementary geometry that PP′, QQ′ meet at a point T on RR′ (Figs. 65, 66). When P′Q′

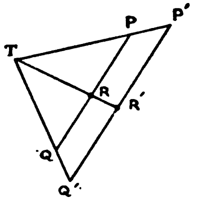

Fig. 65.

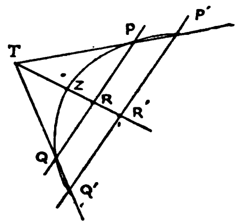

Fig. 66.

moves up to PQ, PP′ becomes the tangent at P, QQ′ the tangent at Q; hence the tangents at P, Q meet on the diameter that bisects PQ. Again, if P moves up to Z, Q also moves up to Z (since R bisects PQ); hence the line PQ becomes the tangent at Z.

112. The methods used for finding the point of bisection of a given chord apply, with very little change, to finding a chord that shall be bisected at a given point.

Example.—Find a chord of $x^2 + 4y^2 = 100$ that shall be bisected at (4, 3).

First solution.—If m is the slope, the equation of the chord is—

$$y - 3 = m(x - 4).$$

The equation for intersections is—

$$x^2 + 4(mx - 4m + 3)^2 = 100,$$

that is, $(4m^2 + 1)x^2 - 8m(4m - 3)x + 4(4m - 3)^2 - 100 = 0.$

The abscissa of the point of bisection is $\dfrac{4m(4m - 3)}{4m^2 + 1}$; the chord is to be chosen so that this shall be 4. Hence—

$$\frac{4m(4m - 3)}{4m^2 + 1} = 4,$$

that is,
$$4m^2 - 3m = 4m^2 + 1,$$
$$m = -\frac{1}{3}.$$

The chord is therefore
$$y - 3 = -\frac{1}{3}(x - 4),$$

that is,
$$x + 3y - 13 = 0.$$

Second solution.—If one extremity of the chord is $(4 + t, 3 + mt)$, the other is $(4 - t, 3 - mt)$. Hence—

$$(4 + t)^2 + 4(3 + mt)^2 = 100,$$
$$(4 - t)^2 + 4(3 - mt)^2 = 100.$$

By subtraction,
$$16t + 48mt = 0,$$
$$m = -\frac{1}{3}, \text{ as before.}$$

The second solution is the simpler. This method will now be applied to find a chord of $\dfrac{x^2}{a} + \dfrac{y^2}{\beta} = 1$ that shall be bisected at (x', y').

Take one extremity of the chord as $(x' + t, y' + mt)$, then the other extremity is $(x' - t, y' - mt)$. These two points are on the conic, hence—

$$\beta(x' + t)^2 + a(y' + mt)^2 = a\beta,$$
$$\beta(x' - t)^2 + a(y' - mt)^2 = a\beta.$$

By subtraction,
$$4t\beta x' + 4mtay' = 0,$$

therefore
$$m = -\frac{\beta x'}{ay'}.$$

Hence the equation of the chord, namely,

$$y - y' = m(x - x')$$

becomes
$$y - y' = -\frac{\beta x'}{\alpha y'}(x - x'),$$

that is,
$$\frac{x'}{\alpha}(x - x') + \frac{y'}{\beta}(y - y') = 0.$$

Notice that there is only one chord bisected at any point other than the centre; for only one value is found for m, unless $x' = 0$, $y' = 0$.

EXAMPLES.

1. Find chords of $y^2 = 4x$, $x^2 + y^2 = 25$, $x^2 - 9y^2 = 36$, $xy = 4$, $2x^2 + 5xy + 2y^2 = 1$, that shall be bisected at (1, 3).

2. Prove that one chord of $y^2 = 4px$ is bisected at (x', y'), namely, $y'(y - y') = 2p(x - x')$.

113. If a chord passes through a fixed point, its point of bisection describes a curve, whose equation can be found at once.

Example.—Find the locus of the points of bisection of chords of $2x^2 - y^2 = 1$ that pass through (4, 5).

By the preceding proof, the chord that is bisected at (x', y') is—

$$2x'(x - x') - y'(y - y') = 0.$$

This goes through (4, 5) if

$$2x'(4 - x') - y'(5 - y') = 0.$$

Hence (x', y') is a point on the curve,

$$2x(4 - x) - y(5 - y) = 0,$$

that is, on
$$2x^2 - y^2 - 8x + 5y = 0.$$

EXAMPLES.

1. Find the locus of the points of bisection of chords of $x^2 + 4y^2 = 100$ that pass through the point (3, 2).

2. Find the locus of the points of bisection of chords of $y^2 = 4x$ that pass through (6, 1).

3. Show that the locus of points of bisection of chords of $\frac{x^2}{a} + \frac{y^2}{\beta} = 1$ that pass through (h, k) is $\frac{x(x-h)}{a} + \frac{y(y-k)}{\beta} = 0$.

4. Show that the locus of points of bisection of chords of $y^2 = 4px$ that pass through (h, k) is $y(y-k) = 2p(x-h)$.

5. Find the locus of the point of bisection of the part of the tangent to $\frac{x^2}{a} + \frac{y^2}{\beta} = 1$ that is included between the axes.

6. Show that the intercept made on a tangent to $\frac{x^2}{a} + \frac{y^2}{\beta} = 1$ by $\frac{x^2}{a} + \frac{y^2}{\beta} = k^2$ is bisected at the point of contact.

7. Prove that the part of the tangent to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ that is included between the asymptotes is bisected at the point of contact.

8. Show that the normal to $2xy = k$ at any point P is the chord of $x^2 - y^2 = a^2$ that is bisected at P.

114. It has been proved that chords of $\frac{x^2}{a} + \frac{y^2}{\beta} = 1$ with the slope m_1 are bisected by a diameter with the slope m_2, where $m_2 = -\frac{\beta}{am_1}$, that is, $m_1m_2 = -\frac{\beta}{a}$. From the symmetry of this relation it follows that chords with the slope m_2 are bisected by a diameter with the slope m_1; that is, if one diameter of a central conic bisects chords parallel to a second diameter, the second bisects chords parallel to the first. Diameters thus related are said to be conjugate. The axes are a particular pair of conjugate diameters.

The word conjugate has the same meaning here as in the case of polars; the two diameters are conjugate lines in the sense of § 100; each passes through the pole of the other, this pole now lying at infinity. The whole theory

of diameters is a part of the theory of polars. If a point P lies at infinity, chords through P are parallel; any one of the chords, being divided harmonically by P and p (§ 101), where P is at infinity, is bisected by p (§ 9), hence the locus of points of bisection of parallel chords is a straight line, the polar of the point at infinity which lies on all these parallel chords. If now Q is the point at infinity on p, chords through Q (that is, chords parallel to p) are bisected by the polar of Q, which goes through P,

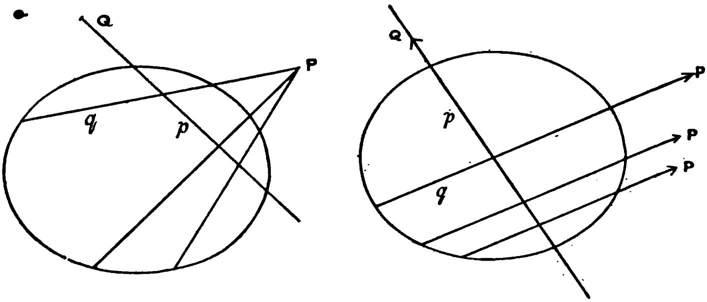

Fig. 67.

since P, Q are conjugate. The conjugate lines p, q are in this case conjugate diameters. Also the chords bisected by a diameter are lines conjugate to the diameter, for they pass through its pole (Fig. 67).

115. The relation that connects the slopes of conjugate diameters of an ellipse is $m_1 m_2 = -\frac{b^2}{a^2}$, hence m_1, m_2 have opposite signs. As one semi-diameter describes the first quadrant, rotating from CA to CB, m_1 is positive and

increases from 0 to ∞; hence m_2 is negative, and decreases numerically from $-\infty$ to -0, and consequently the conjugate semi-diameter describes the second quadrant, rotating in the same direction, from CB to CA′ (Fig. 68).

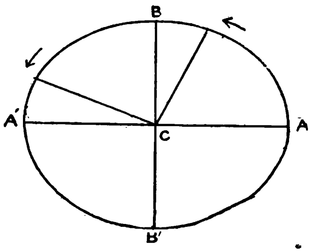

Fig. 68.

A diameter coincides with its conjugate only if $m_2 = m_1$, which for the ellipse gives $m_1{}^2 = -\frac{a^2}{b^2}$, that is, imaginary values for m_1. Hence a diameter of an ellipse, if real, cannot coincide with its conjugate.

Notice also that every real diameter of an ellipse meets the curve in real points.

116. For the hyperbola the relation connecting the

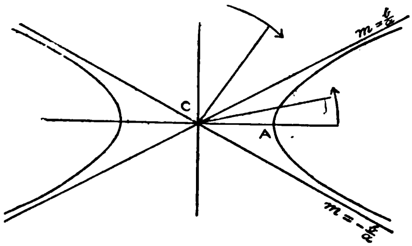

Fig. 69.

slopes of conjugate diameters is $m_1 m_2 = +\frac{b^2}{a^2}$, hence m_1 and m_2 have the same sign. In this case $m_2 = m_1$ when $m_1{}^2 = \frac{b^2}{a^2}$, that is, for two real values of m_1, $+\frac{b}{a}$ and $-\frac{b}{a}$,

which have already been met with as the slopes of the asymptotes. As m_1 increases from 0 to $\frac{b}{a}$, m_2 decreases from ∞ to $\frac{b}{a}$; conjugate diameters approach one another, as shown in Fig. 69.

In Chapter VI. (§ 81) it is shown that a line through C, $y = mx$, meets the hyperbola in real or imaginary points according as m^2 is less than or greater than $\frac{b^2}{a^2}$. But if $m_1 < \frac{b}{a}$, then $m_2 > \frac{b}{a}$, for $m_1 m_2 = \frac{b^2}{a^2}$; and if $m_1 > \frac{b}{a}$, $m_2 < \frac{b}{a}$. Hence of a pair of conjugate diameters, one meets the hyperbola in real points, one in imaginary.

On account of these differences between the ellipse and hyperbola, it is more convenient to consider the two curves separately, at any rate as regards properties of conjugate diameters.

There is no theory of conjugate diameters of a parabola, for a chord parallel to a diameter (which is necessarily itself a diameter) meets the curve in only one finite point, hence there cannot be a point of bisection of such a chord.

117. *Conjugate diameters of the ellipse.*—Of the two diameters, one lies in the first and third quadrants; call this one PP′ (Fig. 70), and the other DD′. If P is (x_1, y_1), the slope of the diameter CP is $m_1 = \frac{y_1}{x_1}$, and the equation of CP is $y = \frac{y_1}{x_1} . x$. If D is (x_2, y_2), the slope of the diameter

CD is $m_2 = \frac{y_2}{x_2}$. Since $m_1 m_2 = -\frac{b^2}{a^2}$, these values for m_1, m_2 give $\frac{y_1 y_2}{x_1 x_2} = -\frac{b^2}{a^2}$, that is, $\frac{x_1 x_2}{a^2} + \frac{y_1 y_2}{b^2} = 0$, as a relation that connects the coordinates of extremities of conjugate diameters.

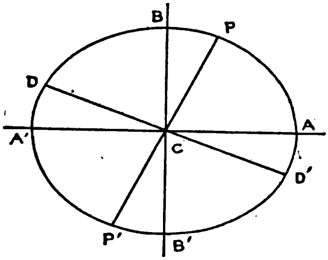

Fig. 70.

The equation of CD, the diameter conjugate to CP, is—

$$y = m_2 x,$$

that is,
$$y = -\frac{b^2}{a^2} \cdot \frac{x_1}{y_1} \cdot x,$$

or
$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 0.$$

This is most easily remembered as the line through the origin parallel to the tangent at P, that is, to

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$$

The points D, D′ are the intersections of CD with the curve; their coordinates are given by—

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 0.$$

Hence
$$\frac{x^2}{a^2} + \frac{b^2x_1^2x^2}{a^4y_1^2} = 1,$$

that is,
$$\frac{x^2}{a^2}\left(1 + \frac{b^2x_1^2}{a^2y_1^2}\right) = 1,$$

or
$$\frac{x^2}{a^2}\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}\right) = \frac{y_1^2}{b^2}.$$

Now P is on the curve, hence $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$, and this equation reduces to—

$$\frac{x^2}{a^2} = \frac{y_1^2}{b^2};$$

therefore
$$\frac{x}{a} = +\frac{y_1}{b} \text{ or } -\frac{y_1}{b},$$

from which, since $\frac{yy_1}{b^2} = -\frac{xx_1}{a^2}$,

$$\frac{y}{b} = -\frac{x_1}{a} \text{ or } +\frac{x_1}{a}.$$

Since the line CD has been taken in the second quadrant, the x of D is negative; hence the point D is $\left(-\frac{ay_1}{b}, +\frac{bx_1}{a}\right)$, and D′ is $\left(\frac{ay_1}{b}, -\frac{bx_1}{a}\right)$. The relation between the coordinates of $P(x_1, y_1)$ or $P'(-x_1, -y_1)$ and $D(x_2, y_2)$ or $D'(-x_2, -y_2)$ is most easily remembered in the form,

$$\frac{x_1}{a} = \pm\frac{y_2}{b}, \quad \frac{x_2}{a} = \pm\frac{y_1}{b},$$

that is, the abscissa of an extremity of one diameter is to a as the ordinate of an extremity of the other diameter is to b, with signs properly adjusted.

For example, the point $\left(4, \frac{12}{5}\right)$ lies on the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$. One extremity of the diameter conjugate to the diameter through this point is $\left(-3, \frac{16}{5}\right)$. The two diameters are $y = \frac{3}{5}x$ and $y = -\frac{16}{15}x$; their slopes satisfy the relation $m_1m_2 = \frac{3}{5} \times -\frac{16}{15} = -\frac{16}{25} = -\frac{b^2}{a^2}$.

EXAMPLES.

1. If the point P on the curve $x^2 + 4y^2 = 100$ is (8, 3), find the coordinates of D. Also if P is (6, 4), find the coordinates of D.

2. If the point P on the curve $\frac{x^2}{50} + \frac{y^2}{32} = 1$ is (5, 4), find the coordinates of D. Find also the equations of the conjugate diameters CP, CD.

3. Prove that if $x_1 = \lambda a$, $y_1 = \mu b$, then $x_2 = -\mu a$, $y_2 = \lambda b$.

118. From the values found for x_2, y_2 in terms of x_1, y_1, we find—

$$x_1^2 + x_2^2 = x_1^2 + \frac{a^2y_1^2}{b^2} = a^2\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}\right) = a^2,$$

$$y_1^2 + y_2^2 = y_1^2 + \frac{b^2x_1^2}{a^2} = b^2\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}\right) = b^2.$$

Now

$$CP^2 = x_1^2 + y_1^2,$$

$$CD^2 = x_2^2 + y_2^2,$$

hence

$$CP^2 + CD^2 = x_1^2 + x_2^2 + y_1^2 + y_2^2 = a^2 + b^2,$$

that is, the sum of the squares of conjugate diameters of an ellipse is constant, $= 4(a^2 + b^2)$.

119. Conjugate diameters are ordinarily of different lengths, for

$$\begin{aligned}
\mathrm{CP}^2 - \mathrm{CD}^2 &= x_1{}^2 - x_2{}^2 + y_1{}^2 - y_2{}^2 \\
&= x_1{}^2 - \frac{a^2y_1{}^2}{b^2} + y_1{}^2 - \frac{b^2x_1{}^2}{a^2} \\
&= x_1{}^2\left(1 - \frac{b^2}{a^2}\right) - y_1{}^2\left(\frac{a^2}{b^2} - 1\right) \\
&= \frac{x_1{}^2}{a^2}\left(a^2 - b^2\right) - \frac{y_1{}^2}{b^2}\left(a^2 - b^2\right) \\
&= \left(\frac{x_1{}^2}{a^2} - \frac{y_1{}^2}{b^2}\right)\left(a^2 - b^2\right).
\end{aligned}$$

Unless the ellipse considered is a circle, when $a = b$, this difference vanishes only if $\frac{x_1{}^2}{a^2} = \frac{y_1{}^2}{b^2}$. Now $\frac{x_1{}^2}{a^2} + \frac{y_1{}^2}{b^2} = 1$, hence in this case $\frac{x_1{}^2}{a^2} = \frac{1}{2}$, $\frac{y_1{}^2}{b^2} = \frac{1}{2}$; consequently P is $\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$, and D is $\left(\frac{-a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$.

These diameters, being both conjugate and equal, are called equiconjugate diameters.

EXAMPLES.

1. Find the equations of the equiconjugate diameters. Prove that they lie along the diagonals of the rectangle formed by the tangents at the vertices.

2. Find the intercepts made on the axes by the tangents at the extremities of the equiconjugate diameters.

3. Find the tangent of the angle that CD makes with CP (i) in terms of the slope of CP, (ii) in terms of the coordinates of P. Find what this becomes if CP, CD are the equiconjugate diameters.

4. The ordinates of P, D are NP, HD. Prove that the triangles CNP, CDH are equal in area.

5. If P is $(\lambda a, \mu b)$, find the equations of the diagonals of the parallelogram formed by the tangents at P, P′, D, D′. Prove that these diagonals lie along conjugate diameters QQ′, EE′.

6. Prove that the diagonals of the parallelogram formed by the tangents at Q, Q′, E, E′ in Ex. 5 lie along PP′, DD′.

7. Prove that the chords that join any point on a central conic to the extremities of a diameter are parallel to a pair of conjugate diameters.

8. Prove, for any central conic, that if CP is conjugate to the normal at Q, then CQ is conjugate to the normal at P.

120. *Conjugate diameters of the hyperbola.*—Of the two diameters, one meets the curve in real points P, P′; take P as (x_1, y_1). Then

$$m_1 = \frac{y_1}{x_1},\ m_1 m_2 = \frac{b^2}{a^2},\ \therefore m_2 = \frac{b^2 x_1}{a^2 y_1}.$$

But $$m_2 = \frac{y_2}{x_2},$$

hence $$\frac{b^2 x_1}{a^2 y_1} = \frac{y_2}{x_2},$$

that is, $$\frac{x_1 x_2}{a^2} - \frac{y_1 y_2}{b^2} = 0$$

is a relation that connects the coordinates of extremities of conjugate diameters.

The diameter conjugate to CP is $y = \dfrac{b^2 x_1}{a^2 y_1} x$, that is—

$$\frac{x x_1}{a^2} - \frac{y y_1}{b^2} = 0,$$

which is the line through the origin parallel to $\dfrac{x x_1}{a^2} - \dfrac{y y_1}{b^2} = 1$, the tangent at P.

This line meets the curve at the points D, D′; the coordinates of these points are therefore given by—

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 0.$$

Hence
$$\frac{x^2}{a^2} - \frac{b^2{x_1}^2x^2}{a^4{y_1}^2} = 1,$$

that is,
$$\frac{x^2}{a^2}\left(\frac{{y_1}^2}{b^2} - \frac{{x_1}^2}{a^2}\right) = \frac{{y_1}^2}{b^2}.$$

Now $\frac{{x_1}^2}{a^2} - \frac{{y_1}^2}{b^2} = 1$, since P is on the curve; this equation therefore becomes—

$$\frac{x^2}{a^2} = -\frac{{y_1}^2}{b^2},$$

that is,
$$\frac{x}{a} = \pm\frac{y_1}{b}i;$$

and
$$y = \frac{b^2x_1}{a^2y_1}x,$$

hence
$$\frac{y}{b} = \pm\frac{x_1}{a}i.$$

The points D, D′ are therefore $\left(\frac{ay_1}{b}i,\ \frac{bx_1}{a}i\right)$ and $\left(-\frac{ay_1}{b}i,\ -\frac{bx_1}{a}i\right)$.

121. The imaginary values found for x_2, y_2 lead to relations for the hyperbola analogous to those found in § 118 for the ellipse, the only difference being in the sign of b^2. We have—

$$x_1^2 + x_2^2 = x_1^2 - \frac{a^2y_1^2}{b^2} = a^2\left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}\right) = a^2,$$

$$y_1^2 + y_2^2 = y_1^2 - \frac{b^2x_1^2}{a^2} = b^2\left(\frac{y_1^2}{b^2} - \frac{x_1^2}{a^2}\right) = -b^2.$$

Now $\quad CP^2 = x_1^2 + y_1^2,$

$\qquad\quad CD^2 = x_2^2 + y_2^2,$

hence

$$CP^2 + CD^2 = x_1^2 + x_2^2 + y_1^2 + y_2^2 = a^2 - b^2.$$

Note.—In each case, $x_1^2 + x_2^2 = \alpha,$

$$y_1^2 + y_2^2 = \beta,$$
$$CP^2 + CD^2 = \alpha + \beta.$$

Thus the algebraic work is the same for the hyperbola as for the ellipse, but it is not susceptible of the same geometrical interpretation. For the ellipse, CD is the length of the semi-diameter conjugate to CP; for the hyperbola, CD is imaginary, it is not a *length* in the usual application of the term. This matter will be taken up again in Chapter IX.

122. *Example* i.—Find the locus of the point of intersection of normals to an ellipse at P, D.

The normal at P is $\frac{a^2(x - x_1)}{x_1} - \frac{b^2(y - y_1)}{y_1} = 0,$

that is,
$$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2.$$

The normal at D is
$$\frac{a^2x}{x_2} - \frac{b^2y}{y_2} = a^2 - b^2.$$

Since P, D are conjugate, $\frac{x_2}{a} = -\frac{y_1}{b},\ \frac{y_2}{b} = \frac{x_1}{a},$

hence the normal at D becomes—

$$-\frac{b}{y_1} \cdot ax - \frac{a}{x_1} \cdot by = a^2 - b^2,$$

that is,
$$\frac{x}{y_1} + \frac{y}{x_1} = -\frac{a^2 - b^2}{ab}.$$

The problem is now reduced to the following :—Find the locus of a point (x, y) determined by the equations—

$$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{x}{y_1} + \frac{y}{x_1} = -\frac{a^2 - b^2}{ab}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where x_1, y_1 are parameters connected by the relation—

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

To obtain the result, eliminate x_1, y_1 from the three equations; the easiest way is to solve the first two for $\frac{1}{x_1}$, $\frac{1}{y_1}$, since these quantities are involved in the first degree. This gives—

$$\frac{1}{x_1}(a^2x^2 + b^2y^2) = (a^2 - b^2)\left(x - \frac{by}{a}\right);$$

$$\frac{1}{y_1}(a^2x^2 + b^2y^2) = (a^2 - b^2)\left(-y - \frac{ax}{b}\right),$$

hence $$\frac{x_1}{a} = \frac{a^2x^2 + b^2y^2}{(a^2 - b^2)(ax - by)}, \; \frac{y_1}{b} = -\frac{a^2x^2 + b^2y^2}{(a^2 - b^2)(ax + by)}.$$

These values, substituted in the third equation, give the result—

$$\frac{(a^2x^2 + b^2y^2)^2}{(a^2 - b^2)^2}\left[\frac{1}{(ax - by)^2} + \frac{1}{(ax + by)^2}\right] = 1,$$

that is, $$2(a^2x^2 + b^2y^2)^3 = (a^2 - b^2)^2(a^2x^2 - b^2y^2)^2.$$

Example ii.—CP, CQ are semi-diameters of an ellipse, in the same quadrant; CD, CE are the semi-diameters conjugate to these, also in one quadrant. Show that the chords PQ, DE are parallel to conjugate diameters.

To prove this result, it is necessary to form the equations of the two chords, and then to show that their slopes satisfy $m_1m_2 = -\frac{b^2}{a^2}$.

If $$\text{P is } (x_1, y_1), \text{ D is } \left(\frac{-ay_1}{b}, \frac{bx_1}{a}\right);$$

and if $$\text{Q is } (x_2, y_2), \text{ E is } \left(\frac{-ay_2}{b}, \frac{bx_2}{a}\right).$$

Hence the equation of PQ is—

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2},$$

and the slope of PQ is $$m_1 = \frac{y_1 - y_2}{x_1 - x_2}.$$

The equation of DE is—

$$\frac{x + \frac{ay_1}{b}}{-\frac{ay_1}{b} + \frac{ay_2}{b}} = \frac{y - \frac{bx_1}{a}}{\frac{bx_1}{a} - \frac{bx_2}{a}},$$

the slope is therefore—

$$m_2 = \frac{\frac{b}{a}(x_1 - x_2)}{-\frac{a}{b}(y_1 - y_2)} = -\frac{b^2}{a^2} \cdot \frac{x_1 - x_2}{y_1 - y_2}.$$

Hence $$m_1 m_2 = \frac{y_1 - y_2}{x_1 - x_2} \times -\frac{b^2}{a_2} \cdot \frac{x_1 - x_2}{y_1 - y_2}$$
$$= -\frac{b^2}{a^2};$$

the two chords PQ, DE are therefore parallel to conjugate diameters.

Examples.

1. Find the locus of the point of bisection of a chord which joins the extremities of conjugate diameters of an ellipse. Interpret the algebraic result.

2. Find the envelope of a chord which joins the extremities of conjugate diameters of an ellipse. Find also the point-equation of this curve. Interpret the algebraic result.

3. Prove that the two lines,

$$\left(\frac{x}{a} + \frac{y}{b}\right) + \lambda\left(\frac{x}{a} - \frac{y}{b}\right) = 0,$$
$$\left(\frac{x}{a} + \frac{y}{b}\right) + \mu\left(\frac{x}{a} - \frac{y}{b}\right) = 0,$$

are conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2}$ if $\lambda\mu = -1$, and conjugate diameters of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $\lambda + \mu = 0$.

4. Show that the equation of a pair of conjugate diameters can be written $\frac{x^2}{a^2} + 2k.\frac{xy}{ab} - \frac{y^2}{b^2} = 0$, for an ellipse, and $\frac{x^2}{a^2} + 2k\frac{xy}{ab} + \frac{y^2}{b^2} = 0$ for a hyperbola.

Find the values of k that give (i) the equiconjugate diameters of the ellipse, (ii) the asymptotes of the hyperbola.

5. Find a diameter of the conics $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, whose conjugates with respect to the two conics shall be at right angles.

Show that this diameter meets each of the conics on the directrices of the other conic; and that the tangents at the points where this diameter meets either conic pass each through a focus of the other conic.

6. Ellipses are described with the same major axis, but with different minor axes. Show that if the extremities of diameters of the different ellipses lie on a line perpendicular to the major axis, then the extremities of the conjugate diameters also lie on a line perpendicular to the major axis.

CHAPTER IX

ASYMPTOTES

123. In Chapter VI. it was shown that the lines $y = \pm \frac{b}{a}x$ are tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at infinity, and that no *real* lines are tangent to the ellipse at infinity. This difference shows plainly in the form of the curves. The ellipse lies within a rectangle of length $2a$ and breadth $2b$, thus, like the circle, it is confined to a finite part of the plane; the hyperbola stretches out indefinitely, we can never draw the whole. In determining the appearance of any curve, it is necessary to know whether the curve is confined to the finite part of the plane, or extends to infinity; and if it does extend to infinity, in what directions. When we have found that the curve passes off to infinity in a certain direction, we consider also which of all the lines in this direction lies closest to the curve at infinity; that is, we determine the tangent, or asymptote, as it is called.

Definition.—A tangent whose point of contact, but not the whole tangent, lies at infinity, is called an asymptote.

124. To find the directions to infinity, form the equation for intersections of the curve with the line $y = mx + n$,

where m, n are to be chosen so as to make the line meet the curve at infinity. It will be found that for a proper choice of m, all the parallel lines $y = mx + n$ will meet the curve in one point at infinity; and that one particular value of n, dependent on the value already chosen for m, will determine the line of this system that meets the curve in two points at infinity.

Note.—It is known that one root of the equation $ax^2 + bx + c = 0$ is zero if $c = 0$; and that if also $b = 0$, then both roots are zero. To examine the conditions for infinite roots, write $x = \frac{1}{z}$; the equation becomes—

$$\frac{a}{z^2} + \frac{b}{z} + c = 0,$$

that is,

$$cz^2 + bz + a = 0.$$

The roots of this equation in z determine the roots of the given equation in x by means of the relation $x = \frac{1}{z}$. If $a = 0$, one value of z is zero, hence one value of x is infinite; and if moreover $b = 0$, the second value of z is zero, hence the second value of x is infinite. That is, if in the quadratic equation $ax^2 + bx + c = 0$, the first coefficient a is zero (or, becomes indefinitely small), one root is infinite (or, becomes indefinitely great); if the two coefficients a, b are zero (or indefinitely small), both roots are infinite (or indefinitely great). Similarly it is shown in the theory of equations that if $a = 0$, the equation $ax^n + bx^{n-1} + cx^{n-2} + \ldots = 0$ has one infinite root; and if also $b = 0$, a second root is infinite.

The equation for intersections of $y = mx + n$ with $\frac{x^2}{\alpha} + \frac{y^2}{\beta} = 1$ is—

$$(\alpha m^2 + \beta)x^2 + 2\alpha mnx + \alpha(n^2 - \beta) = 0.$$

One root is infinite if $\alpha m^2 + \beta = 0$, that is, if $m^2 = -\frac{\beta}{\alpha}$.

For the ellipse, this is $m^2 = -\frac{b^2}{a^2}$, hence m is imaginary; for the hyperbola, $m^2 = +\frac{b^2}{a^2}$, hence m has one of two real values, $\pm\frac{b}{a}$. The ellipse goes to infinity only in two imaginary directions, the hyperbola in two real directions.

All lines with the slope m, determined by $\alpha m^2 + \beta = 0$, have one intersection with the curve at infinity; the second intersection is given by $x = -\frac{n^2-\beta}{2mn}$, $y = \frac{n^2+\beta}{2n}$.

EXAMPLE.

Find where the lines $y = 2x - 5$, $y = -2x - 7$, meet the hyperbola $4x^2 - y^2 = 1$.

Thus the second intersection lies at a finite distance unless $n = 0$. This condition is the one supplied by the vanishing of the second coefficient in the equation for intersections, αmn; for since neither α nor m is zero, it follows that $n = 0$. Hence the lines $y = \pm\frac{b}{a}x$, that is, $\frac{x}{a} \pm \frac{y}{b} = 0$, touch[1] the hyperbola at infinity; they are therefore the asymptotes.

If the equation of the curve be written $\frac{x^2}{\alpha} + \frac{y^2}{\beta} = 1$, either asymptote is $y = mx$, where $m^2 = -\frac{\beta}{\alpha}$.

[1] Distinguish carefully between the words *meet* and *touch*. All lines through a point P on a curve meet the curve at P; of these only the tangent at P touches the curve at P; all other lines through P cut the curve at P.

Hence $$\frac{y^2}{x^2} = -\frac{\beta}{\alpha},$$

that is, $$\frac{x^2}{\alpha} + \frac{y^2}{\beta} = 0,$$

is an equation satisfied by any point on either asymptote, it is the equation of the asymptotes.

125. The parabola $y^2 = 4px$ and the line $y = mx + n$ meet where $(mx + n)^2 = 4px$, that is, where

$$m^2x^2 + 2(mn - 2p)x + n^2 = 0.$$

One root of this equation is infinite if $m = 0$, that is, if the line is parallel to the axis of x. All lines parallel to the axis of x, that is, all diameters, meet the curve at infinity; any diameter $y = n$ meets the curve also where $x = \frac{n^2}{4p}$. This second intersection lies at infinity only if n is infinite, a condition which can be obtained also from the vanishing of the second coefficient $mn - 2p$. Thus the only line that meets $y^2 = 4px$ in two points at infinity is $y = \infty$; this line lies entirely at infinity, and so is not an asymptote. The parabola has therefore no asymptotes; it has one tangent that lies entirely at infinity.

126. For the curve $ax^2 + 2hxy + by^2 = c$, the equation for intersections with $y = mx + n$ is—

$$(a + 2hm + bm^2)x^2 + 2(hn + bmn)x + bn^2 - c = 0.$$

The line $y = mx + n$ is an asymptote if

$$a + 2hm + bm^2 = 0, \; hn + bmn = 0.$$

The first of these equations gives two values for m; the second shows that for each of these values of $m, n = 0$. Hence either asymptote is $y = mx$, where m is determined by $a + 2hm + bm^2 = 0$. The elimination of m from these two gives—

$$a + 2h\frac{y}{x} + b\frac{y^2}{x^2} = 0,$$

that is, $$ax^2 + 2hxy + by^2 = 0,$$

as the relation satisfied by the coordinates of any point on an asymptote; the equation of the asymptotes is therefore—

$$ax^2 + 2hxy + by^2 = 0.$$

Since this is a homogeneous equation of the second degree, it represents two straight lines through the origin (§ 64).

The process applies to equations of any degree.

Note.—The equation that determines n is—

$$n(h + bm) = 0,$$

where $$a + 2hm + bm^2 = 0.$$

This gives $n = 0$, unless $h + bm = 0$. But if $h + bm = 0$, then

$$a - \frac{2h^2}{b} + b \, . \, \frac{h^2}{b^2} = 0,$$

that is, $$h^2 = ab;$$

the given equation $ax^2 + 2hxy + by^2 = 1$

is now $$(\sqrt{a}\, . \, x + \sqrt{b}\, . \, y)^2 = 1,$$

and represents two parallel lines, $\sqrt{a}\, . \, x + \sqrt{b}\, . \, y = \pm 1$. In this special case the result of the text does not hold without modification.

Example i.—Find the asymptotes of

$$2x^2 - 5xy + 2y^2 + 7x - 5y + 8 = 0.$$

The line $y = mx + n$ meets this curve where

$$2x^3 - 5x^2(mx+n) + 2x(mx+n)^2 + 7x^2 - 5x(mx+n) + \ldots - 3 = 0,$$

that is, where

$$(2 - 5m + 2m^2)x^3 + (-5n + 4mn + 7 - 5m)x^2 + \ldots - 3 = 0.$$

Hence one intersection is at infinity if $2 - 5m + 2m^2 = 0$, whence $m = 2$ or $\frac{1}{2}$; the second intersection is at infinity if also

$$-5n + 4mn + 7 - 5m = 0,$$

that is, if $$n = \frac{5m - 7}{4m - 5}.$$

If $m = 2$, $n = 1$; if $m = \frac{1}{2}$, $n = \frac{3}{2}$; hence the asymptotes are—

$$y = 2x + 1, \quad y = \frac{1}{2}x + \frac{3}{2}.$$

Their combined equation is—

$$(2x - y + 1)(x - 2y + 3) = 0,$$

that is, $$2x^2 - 5xy + 2y^2 + 7x - 5y + 3 = 0.$$

Example ii. Find the asymptotes of—

$$x^2y + xy^2 - 2y^3 - x^2 - 2xy + 9y^2 + x - 13y + 10 = 0.$$

The equation for intersections is—

$$\begin{aligned}(m + m^2 - 2m^3)x^3 \\ + (n + 2mn - 6m^2n - 1 - 2m + 9m^2)x^2 \\ + (n^2 - 6mn^2 - 2n + 18mn + 1 - 13m)x \\ - 2n^3 + 9n^2 - 13n + 10 = 0.\end{aligned}$$

The directions to infinity are given by—

$$m + m^2 - 2m^3 = 0, \text{ that is, by } m = 0, 1, -\frac{1}{2};$$

a second intersection lies at infinity if also

$$n + 2mn - 6m^2n - 1 - 2m + 9m^2 = 0,$$

that is, if $$n = \frac{9m^2 - 2m - 1}{6m^2 - 2m - 1}.$$

The values $0, 1, -\frac{1}{2}$ for m give for n the values $1, 2, \frac{3}{2}$. Hence the asymptotes are $y = 1$, $y = x + 2$, $y = -\frac{1}{2}x + \frac{3}{2}$.

EXAMPLES.

1. Find the asymptotes of—

$$\text{(i)}\ x^2 + xy - 2y^2 - 6x + 15y - 28 = 0.$$
$$\text{(ii)}\ x^2 + 5xy + 6y^2 + 3x + 5y - 7 = 0.$$
$$\text{(iii)}\ 3x^2 + 14xy - 5y^2 - 7x - 3y - 4 = 0.$$
$$\text{(iv)}\ (px + qy + r)(p'x + q'y + r') = c.$$

2. Show that the axes of coordinates are the asymptotes of $2xy = k$.

3. A point moves so that the product of its distances from two lines, inclined at an angle $2a$, is constant. Show that it describes a hyperbola of which the two lines are the asymptotes. Find the eccentricity of the hyperbola.

4. A point starts from (1, 1), and moves so that the product of its distances from the lines $x = 0$, $y = 0$, $x + y - 1 = 0$ is constant. Find the equation of the locus, and find the asymptotes.

5. A point starts from $\left(1, -\frac{1}{2}\right)$, and moves so that the product of its distances from the axes varies as the square of its distance from the line $x + y - 1 = 0$. Find the equation of the locus. Find also the asymptotes.

6. Find the real asymptotes of—

$$\text{(i)}\ x^4 - y^4 - xy = 0.$$
$$\text{(ii)}\ x^3 + y^3 - 3xy = 0.$$

127. *The hyperbola and its asymptotes.*—Many geometrical properties of the hyperbola depend on its asymptotes. These lines are $y = \pm \frac{b}{a}x$; hence they are the diagonals of the rectangle constructed by means of the major axis and minor axis of the curve (§ 75).

A point P tracing the hyperbola passes through a vertex A in a direction at right angles to the major axis; it then continually approaches one asymptote as it recedes

from A. This depends on the fact that if NP, the ordinate of P, meets the nearer asymptote at p (Fig. 71), the length

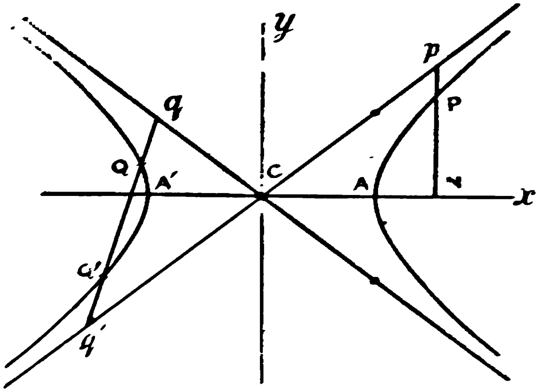

Fig. 71.

Pp continually diminishes as x, that is, CN, increases. Let P be (x, y), and $p(x, y')$, so that $y' = \frac{b}{a}x$.

Then $$Pp = y' - y.$$

Now $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

and $$\frac{x^2}{a^2} - \frac{y'^2}{b^2} = 0,$$

hence, by subtraction, $$\frac{y'^2 - y^2}{b^2} = 1,$$

that is, $$y'^2 - y^2 = b^2,$$

or $$(y' - y)(y' + y) = b^2,$$

from which $$y' - y = \frac{b^2}{y' + y}.$$

Now as x increases, y increases indefinitely, since

$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1$; and since $y' = \frac{b}{a}x$, y' also increases indefinitely; hence as x increases, $y + y'$ increases indefinitely, therefore $\frac{b}{y + y'}$ decreases indefinitely, that is, $y' - y$, which is Pp, decreases indefinitely.

128. The segments of any chord included between the hyperbola and its asymptotes are equal; in Fig. 71, $Qq = Q'q'$. This is shown by means of the theorem, now to be proved, that the point of bisection of the chord of the hyperbola, QQ', is the same as the point of bisection of the chord of the asymptotes, qq'.

The line $y = mx + n$ meets the hyperbola where

$$(a^2m^2 - b^2)x^2 + 2a^2mnx + a^2(n^2 + b^2) = 0.$$

Hence at the point of bisection $\qquad x = -\frac{a^2mn}{a^2m^2 - b^2},$

and therefore $\qquad y = -\frac{b^2n}{a^2m^2 - b^2}.$

Again, the line meets the asymptotes $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$ where

$$(a^2m^2 - b^2)x^2 + 2a^2mnx + a^2n^2 = 0;$$

hence at the point of bisection $\qquad x = -\frac{a^2mn}{a^2m^2 - b^2},$

and therefore $\qquad y = -\frac{b^2n}{a^2m^2 - b^2}.$

These values show that the point of bisection is the same in both cases.

Example.

Make use of the property of the equality of segments to construct points on a hyperbola when the asymptotes and one point P are given.

129. As a particular case of this theorem, let Q′ move up to Q, so that the chord QQ′ becomes the tangent at Q; the theorem then becomes—the part of the tangent that is included between the asymptotes is bisected at the point of contact. An independent proof follows.

The tangent at a point (x_1, y_1) on the hyperbola is—

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$$

This meets one asymptote, $\frac{x}{a} = \frac{y}{b}$, where

$$\frac{xx_1}{a^2} - \frac{y_1}{b} \cdot \frac{x}{a} = 1,$$

that is, $$\frac{x}{a}\left(\frac{x_1}{a} - \frac{y_1}{b}\right) = 1,$$

hence $$\frac{x}{a} = \frac{1}{\frac{x_1}{a} - \frac{y_1}{b}}.$$

Now $$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1,$$

that is, $$\left(\frac{x_1}{a} - \frac{y_1}{b}\right)\left(\frac{x_1}{a} + \frac{y_1}{b}\right) = 1,$$

hence $$\frac{1}{\frac{x_1}{a} - \frac{y_1}{b}} = \frac{x_1}{a} + \frac{y_1}{b}.$$

For the point T accordingly in which the tangent meets the asymptote $\frac{x}{a} = \frac{y}{b}$,

$$x' = a\left(\frac{x_1}{a} + \frac{y_1}{b}\right),$$

$$y' = b\left(\frac{x_1}{a} + \frac{y_1}{b}\right).$$

Similarly for the point T′ in which the tangent meets the asymptote $\frac{x}{a} = -\frac{y}{b}$,

$$x'' = a\left(\frac{x_1}{a} - \frac{y_1}{b}\right),$$

$$y'' = -b\left(\frac{x_1}{a} - \frac{y_1}{b}\right).$$

Hence the coordinates of the point of bisection of TT′ are—

$$x = \frac{x' + x''}{2} = x_1,\ y = \frac{y' + y''}{2} = y_1.$$

130. The example in § 128 shows that a hyperbola is determined when the asymptotes and one point are given. This appears also by the following algebraic proof. Take the axes of coordinates to bisect the angles between the asymptotes; these have then the equations $y = mx$, $y = -mx$. Their combined equation is $m^2x^2 - y^2 = 0$, which takes the form $\frac{x^2}{\alpha} + \frac{y^2}{\beta} = 0$ when $-\frac{\beta}{\alpha}$ is written for m^2.

Any hyperbola with these asymptotes has its axes along the axes of coordinates, hence its equation is $\frac{x^2}{\alpha_1} + \frac{y^2}{\beta_1} = 1$. The asymptotes are $\frac{x^2}{\alpha_1} + \frac{y^2}{\beta_1} = 0$, which

are the same as the given lines if $\dfrac{\alpha_1}{\alpha} = \dfrac{\beta_1}{\beta}$, that is, if $\alpha_1 = k\alpha$, $\beta_1 = k\beta$, where k has any value whatever. Hence any hyperbola with the given asymptotes is—

$$\frac{x^2}{k\alpha} + \frac{y^2}{k\beta} = 1,$$

that is,
$$\frac{x^2}{\alpha} + \frac{y^2}{\beta} = k,$$

and we can choose k to make this pass through any one given point.

If the value of k thus determined is positive, then (for α positive and β negative) the transverse axis lies along the axis of x; but if k is negative, the transverse axis lies along the axis of y.

For example, the hyperbola that has $y = 3x$, $y = -3x$ for asymptotes, and passes through (2, 3) is $9x^2 - y^2 = k$, where $k = 9 \times 4 - 9 = 27$. The hyperbola with these same asymptotes, that passes through (1, 4) is given by $k = 9 - 16 = -7$.

EXAMPLES.

1. Draw the two hyperbolas $9x^2 - y^2 = 27$, $9x^2 - y^2 = -7$ on one diagram.

2. Find the hyperbola with asymptotes $y = \pm 2x$, to pass through (1, 1); also the hyperbola with these same asymptotes to pass through (1, 3). Draw both hyperbolas on one diagram.

131. All hyperbolas with the same asymptotes have the same pairs of conjugate diameters. The hyperbolas are represented by the equation—

$$\frac{x^2}{\alpha} + \frac{y^2}{\beta} = k, \text{ that is, by } \frac{x^2}{k\alpha} + \frac{y^2}{k\beta} = 1,$$

different hyperbolas having different values for k. Two diameters $y = m_1x$, $y = m_2x$, are conjugate with respect to any particular hyperbola—

$$\frac{x^2}{k\alpha} + \frac{y^2}{k\beta} = 1$$

if $$m_1 m_2 = -\frac{k\beta}{k\alpha},$$

that is, if $$m_1 m_2 = -\frac{\beta}{\alpha},$$

a relation which is independent of k, that is, of the particular hyperbola among all that have the given asymptotes. Hence the theorem holds.

132. A diameter is properly an unlimited line through the centre; but when a diameter meets a conic in real points P, P′, the segment PP′ is often spoken of as the diameter. But if the diameter does not meet the curve in real points, a different definition is adopted. It was shown in § 81 that if a diameter, CD, does not meet the curve in real points, its slope $> \frac{b}{a}$, and in § 80 that there is then a real tangent with this slope. The point of contact of this tangent, P, is an extremity of the diameter CP conjugate to CD. The length of the diameter CD conjugate to CP, is defined as equal to that of the part of the tangent at P that is included between the asymptotes, which we know to be bisected at the point of contact. If therefore we draw from T, T′ lines parallel to CP, these determine the extremities of the diameter, as defined above. Call these points d, d' (Fig. 72).

By the construction, the figure CPTd is a parallelogram; hence Pd, CT bisect each other. But P bisects TT′; hence Pd bisects both TT′ and CT, and is therefore

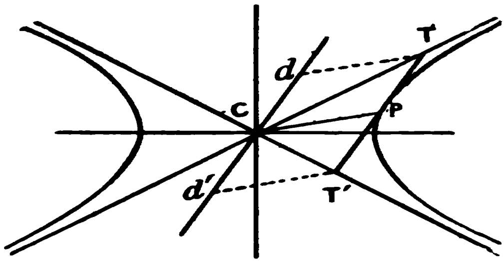

Fig. 72.

parallel to CT′. That is, Pd is parallel to one asymptote, and is bisected by the other. This gives the simplest construction for d.

133. Draw the ordinates of d, P, T, and from P a parallel to the axis of x, meeting the ordinate of T at H (Fig. 73). Since Cd is equal and parallel to PT, the triangles CKd, PHT are equal in all respects. Call the coordinates of d, $x_2'y_2'$. Then

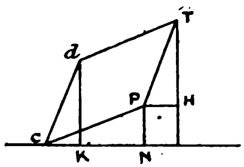

Fig. 73.

$$x_2' = \text{CK} = \text{PH} = x \text{ of T} - x \text{ of P},$$
$$y_2' = \text{K}d = \text{HT} = y \text{ of T} - y \text{ of P}.$$

In § 129 the coordinates of T were found to be—

$$a\left(\frac{x_1}{a} + \frac{y_1}{b}\right),\ b\left(\frac{x_1}{a} + \frac{y_1}{b}\right).$$

Hence $$x_2' = x_1 + \frac{ay_1}{b} - x_1 = \frac{ay_1}{b},$$

$$y_2' = \frac{bx_1}{a} + y_1 - y_1 = \frac{bx_1}{a};$$

that is, the coordinates of d are connected with those of P by the relations $\frac{x_2'}{a} = \frac{y_1}{b}$, $\frac{y_2'}{b} = \frac{x_1}{a}$.

The coordinates of d are connected with the coordinates of D, found in § 120, by a simple relation—

$$x_2 = ix_2',\ y_2 = iy_2'.$$

Also Cd and CD are connected by a corresponding relation;

$$\begin{aligned} \mathrm{CD}^2 &= x_2^2 + y_2^2 = i^2x_2'^2 + i^2y_2'^2 \\ &= -(x_2'^2 + y_2'^2) = -\mathrm{C}d^2, \end{aligned}$$

hence $\mathrm{CD} = i\mathrm{C}d.$

From these relations between D and d, we see that although we cannot actually place the point D in the figure, since it is imaginary, yet we have a memorandum of its position; the x and y of D, and its distance from C, are respectively i times the x and y of d, and its distance from C.

134. As P describes the hyperbola, the point d moves; its path is at once found, for it is the locus of (x_2', y_2'),

where $$x_2' = \frac{ay_1}{b},\ y_2' = \frac{bx_1}{a},$$

and x_1, y_1 are connected by the relation—

$$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1.$$

All that is necessary is to eliminate x_1, y_1.

Write x, y for x_2', y_2'; then

$$x_1 = \frac{ay}{b}, \ y_1 = \frac{bx}{a},$$

consequently the relation connecting x_1, y_1 becomes—

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$$

Hence the point d describes a hyperbola whose equation is—

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1.$$

This is a particular hyperbola of the system determined by $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$ as asymptotes (§ 130).

If we start with a point d on the second hyperbola, the diameter conjugate to Cd is CP, for dP is parallel to one asymptote and bisected by the other. Hence as d describes the second hyperbola, P describes the first. The relation between the two hyperbolas is perfectly symmetrical.

The points P, d are called corresponding points on the two hyperbolas; the point P has two correspondents d, d', and to each of these there correspond the two points P, P′.

Definition.—The hyperbolas—

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = +1, \ \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1,$$

or more generally, the hyperbolas $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm k$, are said to be conjugate.

From the definition it follows that the major and minor

axes of a hyperbola are the minor and major axes of the conjugate hyperbola.

135. By means of the hyperbola conjugate to a given one geometrical properties analogous to those proved for the ellipse in §§ 118, 119 can be formulated.

(i) $$x_1^2 - x_2'^2 = x_1^2 - \frac{a^2y_1^2}{b^2} = a^2\left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}\right) = a^2,$$

$$-y_1^2 + y_2'^2 = -y_1^2 + \frac{b^2x_1^2}{a^2} = b^2\left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}\right) = b^2.$$

(ii) From these by subtraction—

$$\mathrm{CP}^2 - \mathrm{C}d^2 = x_1^2 + y_1^2 - x_2'^2 - y_2'^2 = a^2 - b^2.$$

This last result, stated in words, is the theorem—the difference of the squares of conjugate diameters of a hyperbola (more correctly, of a pair of conjugate hyperbolas) is constant.

136. If 2θ be that angle formed by the asymptotes in which the curve lies, the construction (§ 75) shows that $\tan\theta = \frac{b}{a}$. Hence e^2, which we know to be $\frac{a^2+b^2}{a^2}$, that is, $1 + \frac{b^2}{a^2}$, $= 1 + \tan^2\theta = \sec^2\theta$; that is, $e = \sec\theta$; the eccentricity of a hyperbola is equal to the secant of half the angle formed by the asymptotes.

Example.

Prove that if e, e' are the eccentricities of two conjugate hyperbolas, $\frac{1}{e^2} + \frac{1}{e'^2} = 1$.

137. In the case of the ellipse, the major axis is the greater: this is not necessarily so for a hyperbola. For instance, the hyperbolas—

$$\frac{x^2}{16} - \frac{y^2}{9} = 1, \quad x^2 - y^2 = 16, \quad \frac{x^2}{16} - \frac{y^2}{25} = 1,$$

have the major axis $= 8$, while the minor axis is 6, 8, 10. The construction (§ 75) for the asymptotes of a hyperbola shows that if $a > b$ the hyperbola lies in the pair of acute angles, and if $a < b$ in the pair of obtuse angles formed by the asymptotes; the curve itself is called acute or obtuse. If $a = b$, the asymptotes are at right angles; the curve, whose equation is now $x^2 - y^2 = a^2$, is called a rectangular hyperbola, or sometimes an equilateral hyperbola. The conjugate hyperbola is also rectangular.

Examples.

1. Find the eccentricity of a rectangular hyperbola.

2. Prove that conjugate diameters of a rectangular hyperbola are equal.

138. The equation of any hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, shows that the product of the distances from a point of the curve to the asymptotes is constant; for the asymptotes are $\frac{x}{a} \pm \frac{y}{b} = 0$, and the distances from (x, y) to these lines are

$$\frac{\frac{x}{a} + \frac{y}{b}}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}}, \quad \frac{\frac{x}{a} - \frac{y}{b}}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}},$$ hence the product of the distances is

$\dfrac{\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}}{\dfrac{1}{a^2}+\dfrac{1}{b^2}}$, that is, $\dfrac{a^2b^2}{a^2+b^2}$. If the hyperbola is rectangular, this product is $\dfrac{a^2}{2}$. The hyperbola is therefore the locus of a point which moves so that the product of its distances from two lines is constant. Hence if the asymptotes are $px+qy+r=0$, $p'x+q'y+r'=0$, the equation of the hyperbola is—

$$\frac{(px+qy+r)(p'x+q'y+r')}{\pm\sqrt{p^2+q^2}\sqrt{p'^2+q'^2}}=\text{constant},$$

that is, $(px+qy+r)(p'x+q'y+r')=k$.

The conjugate hyperbola is—

$$(px+qy+r)(p'x+q'y+r')=-k.$$

If the hyperbola is rectangular, the asymptotes may conveniently be taken as axes; the equation is then $xy=k$, where the semi-axis major, a, is determined by $k=\pm\dfrac{a^2}{2}$; hence the equation is better written $2xy=k$, and then $k=\pm a^2$.

If the asymptotes are parallel to the axes, $x-p=0$ and $y-q=0$, the equation is—

$$2(x-p)(y-q)=k,$$

that is, $\quad 2xy-2qx-2py+2pq-k=0,$

which is of the form—

$$2xy+2gx+2fy+c=0.$$

Examples.

1. Prove that any tangent to a hyperbola and the asymptotes include a triangle of constant area.

2. Find the coordinates of the foci, and the equations of the directrices, of the hyperbola conjugate to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

3. Prove that the directrices of the conjugate hyperbolas $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 1$ meet on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and that the foci lie on the director circle of this ellipse.

4. Write down the equations of the tangents at two corresponding points on conjugate hyperbolas. Prove that these tangents meet on an asymptote.

5. Prove that the normals at two corresponding points on conjugate hyperbolas meet on one or other of two lines through the centre perpendicular to the asymptotes.

6. Prove that the polars of any point with respect to conjugate hyperbolas are parallel, and that the centre is half-way between them.

7. Prove that the centre is half-way between the poles of any line with respect to two conjugate hyperbolas.

8. Prove that the pole of any tangent to a hyperbola with respect to the conjugate hyperbola lies on the original hyperbola. How is this pole placed with regard to the point of contact of the tangent?

9. Prove that the normal to $\frac{x^2}{4} - \frac{y^2}{3} = 1$ at the point $\left(4\sqrt{\frac{2}{7}}, \sqrt{\frac{3}{7}}\right)$ is a tangent to the conjugate hyperbola $\frac{x^2}{4} - \frac{y^2}{3} = -1$. Find the point of contact, and show that the normal to the second hyperbola at this point is a tangent to the first hyperbola. Show that the construction can be repeated, so that finally a rectangle is obtained whose vertices lie two on each of the hyperbolas, and whose sides are tangents and normals to the two hyperbolas. Illustrate by a careful diagram.

10. Prove that the part of the tangent to a hyperbola intercepted by any other hyperbola with the same asymptotes is bisected at the point of contact.

11. Find the focus and directrix of the hyperbolas—

$$2xy = k, \quad 2xy + 2gx + 2fy + c = 0.$$

CHAPTER X

PROPERTIES OF CONICS

139. A NUMBER of properties of conics have been proved in Chapters VI.–IX., as illustrations of the different methods there explained. In this chapter these, and other properties, are set forth in order, for the sake of giving a connected view of each conic separately. For some properties, proofs are given; for others, references to sections in which the proofs are to be found. The properties and theorems relating to each conic are numbered for convenience.

I

140. *The parabola. Formulæ and equations.*

Definition.—A parabola is the locus of a point whose distance from a fixed point, the focus, is equal to its distance from a fixed line, the directrix.

Equation.—The simplest form of the equation is $y^2 = 4px$, obtained by taking the origin half-way between the focus and directrix. The focus is $(p, 0)$, the directrix is $x + p = 0$ (§ 70).

If the directrix is $x = q$, and the focus $(q + 2p, 0)$, the equation of the parabola is—

$$\{x - (q + 2p)\}^2 + y^2 = (x - q)^2,$$

that is,

$$\begin{aligned} y^2 &= 4p(x - q) - 4p^2 \\ &= 4px - 4p(p + q). \end{aligned}$$

Hence an equation $y^2 = fx + g$ represents a parabola; the vertex is at $\left(-\frac{g}{f}, 0\right)$, the focus at $\left(-\frac{g}{f} + \frac{f}{4}, 0\right)$, the directrix is $x = -\frac{g}{f} - \frac{f}{4}$.

The latus rectum (§ 70) of the parabola $y^2 = 4px$ is $4p$; of $y^2 = fx + g$ it is f.

The line equation of the parabola is $p\eta^2 = \xi$.

Equations of tangent and normal with a given slope.—The tangent with slope m is $y = mx + \frac{p}{m}$; the point of contact is $\left(\frac{p}{m^2}, \frac{2p}{m}\right)$, often called the point m (§ 79). The normal is $x - \frac{p}{m^2} + m\left(y - \frac{2p}{m}\right) = 0$, that is—

$$y = -\frac{1}{m}x + \frac{2p}{m} + \frac{p}{m^3}.$$

If n is written for the slope, $-\frac{1}{m}$, this becomes—

$$y = nx - 2np - n^3p,$$

which is therefore the equation of the normal with slope n. The point at which the normal is taken is $(n^2p, -2np)$.

Equations of tangent and normal at a given point.—The

tangent at (x_1, y_1) is $yy_1 = 2p(x + x_1)$ (§ 91), the normal at (x_1, y_1) is $y_1(x - x_1) + 2p(y - y_1) = 0$, that is—

$$y = -\frac{y_1}{2p}x + y_1 + \frac{y_1^3}{8p^2}.$$

Equation of chord.—The chord through (x_1, y_1), (x_2, y_2) is $4px - (y_1 + y_2)y + y_1y_2 = 0$; the slope of the chord is $\frac{4p}{y_1 + y_2}$ (§ 91).

The chord through the points m_1, m_2 is—

$$2m_1m_2x - (m_1 + m_2)y + 2p = 0.$$

Focal chord.—The chord (x_1, y_1), (x_2, y_2) passes through the focus $(p, 0)$ if $4p^2 + y_1y_2 = 0$, that is, if $y_1y_2 = -4p^2$. The chord m_1, m_2 passes through the focus if $m_1m_2 = -1$.

Equation of diameter.—Chords of slope m are bisected by the diameter $y = \frac{2p}{m}$ (§ 108).

Pole and polar.—The polar of (x', y') is $yy' = 2p(x + x')$ (§ 97).

Intersection of tangents at two points.—If (x_1, y_1), (x_2, y_2) are the points of contact of the tangents from (x', y'), then $x' = \frac{y_1y_2}{4p}$, $y' = \frac{y_1 + y_2}{2}$. To obtain these values, solve for x', y' the two equations,

$$y_1y' = 2p(x' + x_1),$$
$$y_2y' = 2p(x' + x_2),$$

which express that the tangents at (x_1, y_1), (x_2, y_2), pass through (x', y').

For the points m_1, m_2 these become—

$$x' = \frac{p}{m_1 m_2}, \quad y' = \frac{p}{m_1} + \frac{p}{m_2}.$$

Intersection of normals at two points.—The normals at (x_1, y_1), (x_2, y_2) are—

$$y = -\frac{y_1}{2p}x + y_1 + \frac{y_1^3}{8p^2},$$

$$y = -\frac{y_2}{2p}x + y_2 + \frac{y_2^3}{8p^2}.$$

Their intersection is obtained by solving these equations. Hence the normals at (x_1, y_1), (x_2, y_2) intersect at (x'', y'')

where
$$x'' = \frac{y_1^2 + y_1 y_2 + y_2^2}{4p} + 2p,$$

$$y'' = -\frac{y_1 y_2 (y_1 + y_2)}{8p^2}.$$

The value for x'' can be written $\dfrac{(y_1 + y_2)^2 - y_1 y_2}{4p} + 2p$.

Note.—All formulæ and equations that involve the coordinates of points on the parabola can be written so as to involve the ordinates only, by means of the relation $x_1 = \dfrac{y_1^2}{4p}$.

The coordinates of the intersection of the tangents, or normals, at two points involve only the sum and product of the ordinates.

141. *The parabola. Properties and theorems.*

(i) By definition, the focal distance—

$$\begin{aligned} SP &= MP \\ &= XN \text{ (Fig. 74)} \\ &= AN + XA \\ &= x_1 + p. \end{aligned}$$

(ii) From the equation $y^2 = 4px$,

$$NP^2 = 4AS \,.\, AN,$$

that is, the square of the ordinate varies as the abscissa.

(iii) The subtangent is bisected at the vertex.

The tangent is $yy_1 = 2p(x + x_1)$,

hence at T, $y = 0, x = -x_1$,

that is, $TA = AN$.

Alternative statement.—The subtangent $= 2 \times$ abscissa.

(iv) Since $TA = AN$, and $AS = XA$,

$$\therefore\ TA + AS = XA + AN,$$

that is, $TS = p + x_1 = SP$.

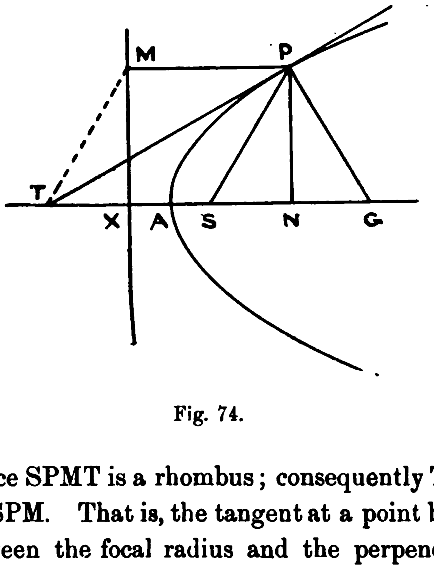

Fig. 74.

(v) Hence SPMT is a rhombus; consequently TP bisects the angle SPM. That is, the tangent at a point bisects the angle between the focal radius and the perpendicular to the directrix.

Alternative statement.—Since TS = SP,

$$\therefore \ \angle \text{STP} = \angle \text{SPT},$$

therefore the tangent is equally inclined to the axis and the focal radius.

(vi) The subnormal is constant, and equal to the semi-latus rectum.

The angle TPG is a right angle, hence SPG is the complement of SPT. Also SGP is the complement of STP.

Since SPT and STP are equal,

$$\angle \text{SGP} = \angle \text{SPG},$$

$$\therefore \ \text{SG} = \text{SP} = \text{XN}.$$

$$\therefore \ \text{SN} + \text{NG} = \text{XS} + \text{SN},$$

hence $\text{NG} = \text{XS} = \frac{1}{2}$ latus rectum.

Alternative proof.—The normal is—

$$y_1(x - x_1) + 2p(y - y_1) = 0.$$

At G, $y = 0$, hence $y_1(x - x_1) - 2py_1 = 0,$

or $x - x_1 = 2p;$

that is, $\text{AG} - \text{AN} = 2p,$

$$\text{NG} = 2p.$$

(vii) Since SPMT is a rhombus, SM and TP bisect one another at right angles. Call the point of intersection Y (Fig. 75). Then A bisects SX and Y bisects SM, hence AY is parallel to XM, and is therefore the tangent at the vertex. That is, the foot of the perpendicular from the focus on any tangent lies on the tangent at the vertex.

Alternative proof.—The tangent is $y = mx + \dfrac{p}{m}$; the

perpendicular from S is $y = -\frac{1}{m}(x-p)$. These meet where $\left(m+\frac{1}{m}\right)x = 0$, that is, on the axis of y.

(viii) $SY^2 = AS.SP$ (Fig. 75).

The tangent is $$yy_1 = 2p(x+x_1),$$
that is, $$2px - y_1y + 2px_1 = 0.$$
Hence the distance from $(p, 0)$ is $\dfrac{2p^2+2px_1}{\sqrt{4p^2+y_1^2}}$.

This gives
$$\begin{aligned} SY^2 &= \frac{(2p^2+2px_1)^2}{4p^2+y_1^2} \\ &= \frac{4p^2(p+x_1)^2}{4p^2+4px_1} \\ &= p(p+x_1) \\ &= AS.SP. \end{aligned}$$

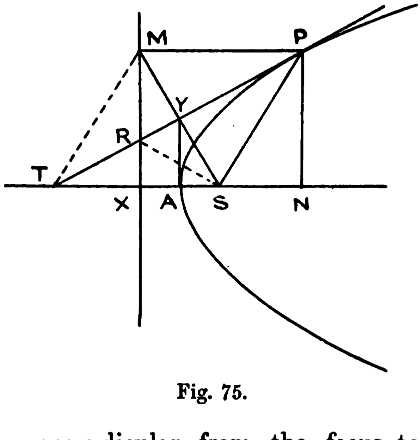

Fig. 75.

(ix) The perpendicular from the focus to any chord meets the directrix on the diameter that bisects the chord.

The chord $y = mx - mp$ is bisected by the diameter $y = \dfrac{2p}{m}$.

The line through S perpendicular to the chord is—

$$y = -\frac{1}{m}(x - p);$$

this meets the directrix, $x = -p$, where

$$y = -\frac{1}{m}(-p - p),$$

that is, where $$y = \frac{2p}{m}.$$

(x) The semi-latus rectum is a harmonic mean between the two segments of a focal chord.

If the extremities of the chord are $P(x_1, y_1)$ and $Q(x_2, y_2)$, $$SP = p + x_1,\ SQ = p + x_2,$$

where $$x_1x_2 = \frac{y_1^2y_2^2}{16p^2} = \frac{16p^4}{16p^2} = p^2.$$

Hence $$\frac{1}{SP} + \frac{1}{SQ} = \frac{1}{p + x_1} + \frac{1}{p + x_2}$$

$$= \frac{2p + x_1 + x_2}{p^2 + px_1 + px_2 + x_1x_2}$$

$$= \frac{2p + x_1 + x_2}{2p^2 + px_1 + px_2}$$

$$= \frac{1}{p}$$

$$= \frac{2}{2p}.$$

Hence $2p$ is a harmonic mean between SP and SQ.

(xi) If the tangent at P meets the directrix at R, then SR is perpendicular to SP (Fig. 75).

The tangent at (x_1, y_1) is $yy_1 = 2p(x + x_1)$, therefore the coordinates of R are $x = -p$,

$$y = \frac{2p(x_1 - p)}{y_1}.$$

Hence the equation of SR is—

$$\frac{x - p}{2p} = \frac{y}{-\dfrac{2p(x_1 - p)}{y_1}},$$

that is, $$\frac{x - p}{y_1} + \frac{y}{x_1 - p} = 0;$$

or $$(x - p)(x_1 - p) + yy_1 = 0.$$

The equation of SP is $\dfrac{x - p}{x_1 - p} = \dfrac{y}{y_1}$,

and these two lines satisfy the condition of perpendicularity.

Alternative statement.—The part of the tangent included between the point of contact and the directrix subtends a right angle at the focus.

(xii) The tangents at the extremities of a focal chord intersect at right angles on the directrix.

The tangents
$$yy_1 = 2p(x + x_1),$$
$$yy_2 = 2p(x + x_2),$$

are perpendicular if $4p^2 + y_1y_2 = 0$, the condition already found to express that the chord passes through the focus.

The point of intersection of the tangents has coordinates $\dfrac{y_1y_2}{4p}, \dfrac{y_1 + y_2}{2}$. Since $y_1y_2 + 4p^2 = 0$, the abscissa is $\dfrac{-4p^2}{4p}$, that is, $-p$; hence the point lies on the directrix.

Otherwise: the condition that the chord m_1, m_2 shall

pass through the focus has been found to be $m_1m_2 = -1$, which is also the condition that the tangents, of slopes m_1, m_2, shall be perpendicular.

Notice that the tangents intersect on the directrix because the directrix is the polar of the focus (§ 98).

(xiii) The locus of the point of intersection of normals at consecutive (indistinguishable) points is—

$$4(x - 2p)^3 = 27py^2.$$

The point of intersection of the normals at (x_1, y_1), (x_2, y_2) is—

$$x'' = \frac{y_1^2 + y_1y_2 + y_2^2}{4p} + 2p,$$

$$y'' = -\frac{y_1y_2(y_1 + y_2)}{8v^2}.$$

Write $\qquad x_2 = x_1,\ y_2 = y_1;$

these become $\qquad x'' = \frac{3y_1^2}{4p} + 2p,\quad y'' = -\frac{2y_1^3}{8p^2}.$

To obtain the locus, eliminate y_1. We have—

$$3y_1^2 = 4p(x'' - 2p),$$

$$2y_1^3 = -8p^2y''.$$

Hence $\qquad 27y_1^6 = 64p^3(x'' - 2p)^3,$

$$4y_1^6 = 64p^4y''^2,$$

from which $4(x'' - 2p)^3 = 27py''^2$.

The locus of (x'', y'') is therefore the curve—

$$4(x - 2p)^3 = 27py^2.$$

Note.—Call the successive normals n_1, n_2, n_3, etc. Then n_1, n_2 meet at a point Q_1, n_2, n_3 at Q_2, n_3, n_4 at Q_3, and so on. When the normals are taken to be indefinitely close together, the points Q_1, Q_2, Q_3, ... become points on a curve; the tangents to this

curve are the lines Q_1Q_2, Q_2Q_3, etc., that is, the normals n_2, n_3. . . . Thus the curve found as the locus of the intersection of consecutive normals is the same as the envelope of the normals. It is called the *evolute* of the parabola or other curve. See p. 362.

(xiv) Three normals to a parabola pass through any point (h, k).

The equation of the normal at (x_1, y_1) is—

$$y = -\frac{y_1}{2p}x + y_1 + \frac{y_1^3}{8p^2};$$

this passes through (h, k) if

$$k = -\frac{y_1}{2p}h + y_1 + \frac{y_1^3}{8p^2}.$$

This is an equation for y_1, the ordinate of the point at which a normal must be taken in order that it may pass through (h, k); y_1 is therefore any root of the equation—

$$k = -\frac{y}{2p}h + y + \frac{y^3}{8p^2},$$

that is, of $$y^3 + 4p(2p - h)y - 8p^2k = 0.$$

Since the equation is of the third degree, there are three possible values for y, call these y_1, y_2, y_3; there are three points, the normals at which pass through (h, k); and since the coefficient of y^2, which by the theory of equations $= -(y_1 + y_2 + y_3)$, is zero, the sum of the ordinates of these three points is zero.

If the normals at points are concurrent, the points may be said to be conormal.

Note.—The cubic equation for y shows that—

$$\begin{aligned} y_1 + y_2 + y_3 &= 0, \\ y_2y_3 + y_3y_1 + y_1y_2 &= 4p(2p - h), \\ y_1y_2y_3 &= 8p^2k, \end{aligned}$$

where h, k are the coordinates of the point common to the three normals. Hence—

$$h = 2p - \frac{y_2y_3 + y_3y_1 + y_1y_2}{4p},$$

$$k = \frac{y_1y_2y_3}{8p^2}.$$

That is, if the sum of the ordinates y_1, y_2, y_3 of three points on a parabola $y^2 = 4px$ is zero, the three normals meet at the point $\left(2p - \frac{\Sigma y_2y_3}{4p}, \frac{y_1y_2y_3}{8p^2}\right)$. An equivalent statement is, the normals at the points whose ordinates are given by $y^3 + fy + g = 0$ meet at the point $\left(2p - \frac{f}{4p}, \frac{-g}{8p^2}\right)$.

(xv) Any circle meets a parabola in four points, the sum of whose ordinates is zero.

The circle $x^2 + y^2 + 2gx + 2fy + c = 0$ meets the parabola $y^2 = 4px$ where

$$\frac{y^4}{16p^2} + y^2 + \frac{2gy^2}{4p} + 2fy + c = 0,$$

that is, where

$$y^4 + 8p(2p + g)y^2 + 32fp^2y + 16p^2c = 0.$$

Since this equation is of the fourth degree, there are four values for y; and since the coefficient of y^3 is zèro, the sum of these four values is zero.

The four points are said to be concyclic.

(xvi) The angles made with the axis by a pair of common chords of a circle and parabola are supplementary.

The chord that joins (x_1, y_1) to (x_2, y_2) has the slope $\frac{4p}{y_1 + y_2}$; the chord that joins the remaining points (x_3, y_3), (x_4, y_4) has the slope $\frac{4p}{y_3 + y_4}$.

But by (xv), $\quad y_1 + y_2 + y_3 + y_4 = 0,$

that is, $\quad y_3 + y_4 = -(y_1 + y_2).$

Hence $\dfrac{4p}{y_3 + y_4} = -\dfrac{4p}{y_1 + y_2}$; that is, if one chord has the slope m, the other has the slope $-m$.

(xvii) The circle through three conormal points passes through the vertex.

Call the ordinates of the three points y_1, y_2, y_3, and let y_4 be the ordinate of the fourth point in which the circle through these three meets the parabola again. Since the four points are concyclic, $y_1 + y_2 + y_3 + y_4 = 0$; and since the three points are conormal, $y_1 + y_2 + y_3 = 0$. Hence $y_4 = 0$, from which $x_4 = \dfrac{y_4^2}{4p} = 0$, and the fourth point is (0, 0), that is, the vertex.

142. *Example* i.—Find the condition satisfied by two points P, Q on the parabola if the area APQ is constant.

The area of the triangle (0, 0), (x_1, y_1), (x_2, y_2) is $\frac{1}{2}(x_1y_2 - x_2y_1)$ (Ex. 19, § 21). Since P, Q are on the parabola, $x_1 = \dfrac{y_1^2}{4p}$, $x_2 = \dfrac{y_2^2}{4p}$; hence the area APQ $= \dfrac{1}{2} \cdot \dfrac{y_1^2y_2 - y_1y_2^2}{4p} = \dfrac{y_1y_2(y_1 - y_2)}{8p} = k$. The condition is therefore, $y_1y_2(y_1 - y_2)$ is constant.

Example ii.—Find the locus of the pole of the chord PQ, if the area APQ is constant.

The coordinates of the pole are $x' = \dfrac{y_1y_2}{4p}$, $y' = \dfrac{y_1 + y_2}{2}$; the condition that the area is constant is (Ex. i)—

$$y_1y_2(y_1 - y_2) = \text{const.} = 8pk.$$

To find the locus of (x', y'), eliminate y_1, y_2. Since $y_1y_2 = 4px'$ and $y_1 + y_2 = 2y'$, therefore—

$$(y_1 - y_2)^2 = (y_1 + y_2)^2 - 4y_1y_2 = 4y'^2 - 16px'.$$

Hence $4px' . \sqrt{4y'^2 - 16px'} = 8pk$ is the equation to be satisfied by x', y'. This shows that the locus of (x', y') is the curve—

$$4px\sqrt{4y^2 - 16px} = 8pk,$$
$$8px\sqrt{y^2 - 4px} = 8pk,$$
$$x^2(y^2 - 4px) = k^2.$$

Example iii.—Find the locus of the point of bisection of a chord PQ, if the area APQ is constant.

Here
$$x' = \frac{x_1 + x_2}{2} = \frac{y_1^2 + y_2^2}{8p},$$
$$y' = \frac{y_1 + y_2}{2};$$

y_1, y_2 are connected by the same relation as before. To eliminate y_1, y_2, find y_1y_2 from these two equations, then $y_1 - y_2$; substitute these in the relation that connects y_1, y_2.

We have $\quad y_1 + y_2 = 2y'$,

and $\quad y_1^2 + y_2^2 = 8px'$,

hence
$$2y_1y_2 = (y_1 + y_2)^2 - (y_1^2 + y_2^2)$$
$$= 4y'^2 - 8px',$$

therefore
$$y_1y_2 = 2y'^2 - 4px' = 2(y'^2 - 2px').$$

Also
$$(y_1 - y_2)^2 = y_1^2 + y_2^2 - 2y_1y_2$$
$$= -4y'^2 + 16px' = -4(y'^2 - 4px').$$

Hence x', y' satisfy—

$$2(y'^2 - 2px') \times 2\sqrt{-y'^2 + 4px'} = 8pk;$$

the locus of (x', y') is therefore the curve—

$$-(y^2 - 2px)^2(y^2 - 4px) = 4p^2k^2,$$

that is,
$$(y^2 - 2px)^2(y^2 - 4px) + 4p^2k^2 = 0.$$

Example iv.—Find the envelope of the chord PQ, if the area APQ is constant.

The equation of the chord is $4px - (y_1 + y_2)y + y_1y_2 = 0$. Hence $\xi = \dfrac{4p}{y_1y_2}$, $\eta = -\dfrac{y_1 + y_2}{y_1y_2}$. From these two equations, combined with $y_1y_2(y_1 - y_2) = 8pk$, we are to eliminate y_1, y_2.

We have
$$y_1 y_2 = \frac{4p}{\xi},$$
$$y_1 + y_2 = -\frac{4p\eta}{\xi},$$
hence
$$(y_1 - y_2)^2 = \frac{16p^2\eta^2}{\xi^2} - \frac{16p}{\xi}$$
$$= \frac{16}{\xi^2}\left(p^2\eta^2 - p\xi\right);$$
the envelope is therefore—
$$\frac{4p}{\xi}\cdot\frac{4}{\xi}\sqrt{p^2\eta^2 - p\xi} = 8pk,$$
that is,
$$2\sqrt{p^2\eta^2 - p\xi} = k\xi^2,$$
or
$$k^2\xi^4 - 4p^2\eta^2 + 4p\xi = 0.$$

Examples on the Parabola.

1. If the product of the abscissæ of two points on a parabola is equal to AS^2, the line that joins the two points passes either through the focus or through the foot of the directrix.

2. If the product of the abscissæ of two points on a parabola has any constant value, the line that joins the two points passes through one or other of two fixed points on the axis which are equidistant from the vertex.

3. The lines that join any point on a parabola to the extremities of a chord perpendicular to the axis meet the axis at two points equidistant from the vertex.

4. If the polar and ordinate of a point H meet the axis at K, L, then KL is bisected at A.

5. If the tangents at points P, P′ on a parabola meet at H, then $SP . SP' = SH^2$.

6. The circle on a focal radius as diameter touches the tangent at the vertex.

7. The circle on a focal chord as diameter touches the directrix.

8. The line that joins the point of intersection of the tangents at the extremities of a focal chord to the point of intersection of the normals is a diameter.

9. The locus of the point of intersection of normals at the extremities of a focal chord is the parabola $y^2 = p(x - 3p)$.

10. The locus of the point of intersection of the normals at points

which lie on a line through the foot of the directrix is the parabola $y^2 = p(x - p)$.

11. Find the locus of the foot of the perpendicular from the focus to a normal.

12. Find the locus of the foot of the perpendicular from the vertex to a normal.

13. The locus of the point of intersection of normals at the extremities of a chord which is fixed in direction is a straight line.

14. If the product of the ordinates of two points on a parabola is equal to $8p^2$, the normals at the points meet on the parabola.

15. The part of the axis that is included between the ordinate of any point and the line through that point perpendicular to its polar is of constant length.

16. Find the locus of the point of intersection of two tangents that make a constant angle.

17. Prove that the difference of the squares of the distances to any tangent from two fixed points on the axis, equidistant from the focus, is constant.

18. If four tangents are drawn to a parabola, the points of bisection of the three lines, formed by joining the intersection of any two of these tangents to the intersection of the remaining two, lie on a diameter.

19. If the four points at which the tangents are drawn (Ex. 18) lie on a circle, the three points of bisection lie on the axis of the parabola.

20. The abscissa of the pole of the line that joins two points on a parabola is a geometric mean between the abscissæ of the points, and the ordinate of the pole is the arithmetic mean between the ordinates of the points.

21. A line moves so that the difference of the squares of its distances from two fixed points is constant. Show that in all its positions it is a tangent to a certain parabola.

22. If the product of the ordinates of the extremities of a chord has any constant value, the chord passes through a fixed point on the axis.

23. Find the locus of the foot of the perpendicular from the vertex to a tangent.

24. Find the condition to which the coordinates of a line must be subject in order that the line may be a normal to a parabola. (That is, find the line-equation of the evolute of a parabola, § 141, xiii.)

25. Find the coordinates of the point common to the three normals drawn at the remaining three intersections of the parabola $y^2 = 4px$ with the circle through the vertex, $x^2 + y^2 + 2gx + 2fy = 0$.

26. Show that the circle through three conormal points is—

$$x^2 + y^2 - (2p + h)x - \frac{k}{2}y = 0,$$

where (h, k) is the point of concurrence of the three normals.

27. The centre of the circle through three conormal points describes a straight line of slope m; show that the point of concurrence of the normals describes a straight line of slope $2m$.

28. Three conormal points are chosen so that the circle through them is of constant radius; find the locus of the point of concurrence.

29. Find the locus of the intersection of the normals at the extremities of a chord which passes through a fixed point on the axis.

30. Find the locus of the point of bisection of a chord through a fixed point.

31. Prove that three points on a parabola are conormal if the centroid of the triangle formed by the points is on the axis.

32. Tangents p, q, r are drawn at the points P, Q, R. Show that the line that joins the centroids of the two triangles pqr, PQR is a diameter of the parabola.

33. If the point of intersection of the normals at P, Q describes the line $y = k$, the point of intersection of the tangents describes the rectangular hyperbola $xy = -pk$.

34. If the point of intersection of the normals at P, Q describes the line $x = h$, the point of intersection of the tangents describes the parabola $y^2 = px + p(h - 2p)$.

35. Show that the normals at the points of contact of the tangents from (x', y') meet at (x'', y''), where

$$x'' = \frac{y'^2 - px'}{p} + 2p, \quad y'' = -\frac{x'y'}{p}.$$

36. Form the equation whose roots are the ordinates of the points of intersection of the tangents at three conormal points, the point of concurrence of the normals being (h, k).

37. Show that the lines that join the vertex to the intersections of the parabola with $x + 3y - 4p = 0$ are at right angles.

38. A chord PQ subtends a right angle at the vertex. Find the condition satisfied by the ordinates of P, Q, and apply this to show that the chord meets the axis at a fixed point.

39. Find the locus of the point of bisection of a chord which subtends a right angle at the vertex.

40. Find the locus of the pole of a chord which subtends a right angle at the vertex.

41. Find the locus of the foot of the perpendicular from the vertex on a chord which subtends a right angle at the vertex.

42. The tangents at P and Q meet at T. Show that the distances from P, T, Q to any other tangent are in G.P.

43. Find the locus of a point whose polars with respect to the two parabolas $y^2 = 4px$, $x^2 = 4qy$ are (i) parallel, (ii) perpendicular.

44. If the ordinates of four points on a parabola are in A.P., the distances between the ordinates are also in A.P.

45. The normals at the extremities (x_1, y_1), (x_2, y_2) of a focal chord P_1P_2 meet the curve again at the points $Q_1(x_1', y_1')$, $Q_2(x_2', y_2')$. Show that $y_1' + y_2' = y_1 + y_2$, and hence that Q_1Q_2 is parallel to P_1P_2.

II

143. *The ellipse. Formulæ and equations.*

Definition.—An ellipse is the locus of a point whose distance from a fixed point, the focus, is in a constant ratio, less than unity, to its distance from a fixed line, the directrix. The constant ratio is called the eccentricity.

Equation.—The simplest form of the equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, obtained by taking for the origin the point that divides the distance from the focus to the directrix externally in the ratio $e^2:1$, where e is the eccentricity (§ 71). The symmetry of this form shows that there are two foci, each with its own directrix. The foci are $(\pm ae, 0)$; the corresponding directrices are $x = \pm \frac{a}{e}$. For ae, c is often written; $c^2 = a^2 - b^2$, $e = \frac{\sqrt{a^2 - b^2}}{a}$.

The length of the latus rectum (§ 73) is $2\frac{b^2}{a}$; it lies on the line $x = ae$.

The line-equation of the ellipse is $a^2\xi^2 + b^2\eta^2 = 1$.

Equation of tangent with a given slope.—The tangent with slope m is $y = mx \pm \sqrt{a^2m^2 + b^2}$; the point of contact is $\left(\frac{\mp a^2m}{\sqrt{a^2m^2 + b^2}}, \frac{\pm b^2}{\sqrt{a^2m^2 + b^2}}\right)$ (§ 79).

Equations of tangent and normal at a given point.—The tangent at (x_1, y_1) is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ (§ 91); the normal at (x_1, y_1) is $\frac{a^2}{x_1}(x - x_1) - \frac{b^2}{y_1}(y - y_1) = 0$, that is—

$$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2.$$

Equation of chord.—The chord through (x_1, y_1), (x_2, y_2) is $\frac{x(x_1 + x_2)}{a^2} + \frac{y(y_1 + y_2)}{b^2} = \frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} + 1$; the slope of the chord is $-\frac{b^2(x_1 + x_2)}{a^2(y_1 + y_2)}$ (§ 91).

Pole and polar.—The polar of (x', y') is $\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1$ (§ 97).

Equation of diameter.—Chords of slope m_1 are bisected by the diameter $y = m_2x$, where $m_1m_2 = -\frac{b^2}{a^2}$ (§ 108).

If the extremities of the diameters of slopes m_1, m_2 $\left(\text{where } m_1m_2 = -\frac{b^2}{a^2}\right)$ are (x_1, y_1), (x_2, y_2), then $\frac{y_2}{b} = \pm\frac{x_1}{a}$, and $\frac{x_2}{a} = \mp\frac{y_1}{b}$ (§ 117).

If a semi-diameter CP of length r makes with the axis of x an angle θ, the coordinates of P are $r\cos\theta$, $r\sin\theta$ (§ 22). Hence—

$$\frac{r^2\cos^2\theta}{a^2}+\frac{r^2\sin^2\theta}{b^2}=1,$$

that is, $$r^2\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)=1.$$

From this $$r^2\left(\frac{\cos^2\theta}{a^2}+\frac{1-\cos^2\theta}{b^2}\right)=1,$$

hence $$r^2(a^2-(a^2-b^2)\cos^2\theta)=a^2b^2,$$

that is, $$r^2(1-e^2\cos^2\theta)=b^2.$$

The auxiliary circle and eccentric angle.—Many properties of the ellipse are conveniently proved by means of the relation between the ellipse and the circle on the major axis as diameter, the *auxiliary circle.* The equation of this circle is $x^2+y^2=a^2$.

If the ordinate NP of a point P on the ellipse is produced to meet the auxiliary circle at p, then P, p are called corresponding points (Fig. 76). From the equations of the ellipse and circle,

$$\text{NP}^2=b^2\left(1-\frac{x^2}{a^2}\right),\ \text{N}p^2=a^2\left(1-\frac{x^2}{a^2}\right).$$

Hence $\dfrac{\text{NP}^2}{\text{N}p^2}=\dfrac{b^2}{a^2}$; and since by construction NP and Np have the same sign, $\dfrac{\text{NP}}{\text{N}p}=\dfrac{b}{a}$. That is, the ordinates of corresponding points on the ellipse and the auxiliary circle are in a constant ratio. The position of P is therefore known when that of p is known, and this depends only on

the angle ACp. If this angle is called ϕ, then CN $= a \cos \phi$, Np $= a \sin \phi$, from which NP $= \dfrac{b}{a}$Np $= b \sin \phi$. Hence the coordinates of a point on the ellipse can be expressed

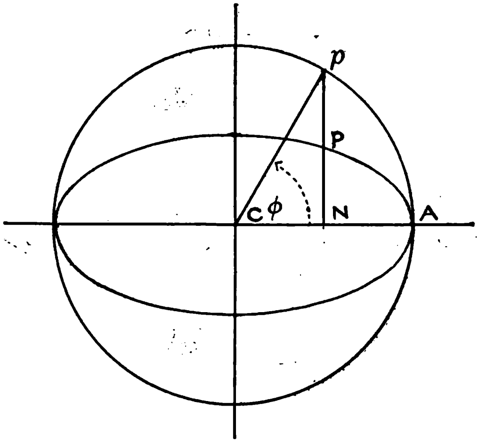

Fig. 76.

as $(a \cos \phi, b \sin \phi)$, where ϕ is the angle ACp. This is called the eccentric angle of the point P on the ellipse. The point P is spoken of as the point ϕ.

Equation of chord in terms of the eccentric angle.—To obtain the chord ϕ_1, ϕ_2 in the most convenient form, write $\alpha = \dfrac{\phi_1 + \phi_2}{2}$, $\beta = \dfrac{\phi_1 - \phi_2}{2}$; the two points are then $\alpha \pm \beta$. The equation of the chord is—

$$\frac{x - a\cos(\alpha+\beta)}{a[\cos(\alpha+\beta) - \cos(\alpha-\beta)]} = \frac{y - b\sin(\alpha+\beta)}{b[\sin(\alpha+\beta) - \sin(\alpha-\beta)]},$$

$$\frac{x - a\cos(\alpha + \beta)}{-2a\sin\alpha\sin\beta} = \frac{y - b\sin(\alpha + \beta)}{2b\cos\alpha\sin\beta},$$

$$\frac{x - a\cos(\alpha + \beta)}{a\sin\alpha} + \frac{y - b\sin(\alpha + \beta)}{b\cos\alpha} = 0,$$

$$\frac{x\cos\alpha}{a} + \frac{y\sin\alpha}{b} = \cos\alpha\cos(\alpha + \beta) + \sin\alpha\sin(\alpha + \beta)$$
$$= \cos\beta.$$

Equation of tangent and normal in terms of the eccentric angle.—The extremities of a chord become indistinguishable if $\alpha + \beta = \alpha - \beta$; that is, if $\beta = 0$. Hence the equation of the tangent is obtained by the substitution of the value 0 for β in the equation of the chord; it is therefore $\quad \dfrac{x\cos\alpha}{a} + \dfrac{y\sin\alpha}{b} = 1.$

The normal is $\dfrac{ax}{\cos\alpha} - \dfrac{by}{\sin\alpha} = a^2 - b^2.$

Diameter in terms of eccentric angle.—If the extremities of a chord are $\alpha \pm \beta$, the equation is—

$$\frac{x\cos\alpha}{a} + \frac{y\sin\alpha}{b} = \cos\beta,$$

the slope is therefore $-\dfrac{b\cos\alpha}{a\sin\alpha}$. This is constant if α is constant, hence all chords with this same slope are obtained by varying β. In particular, the tangent is given by $\beta = 0$; the diameter (line through origin) by $\cos\beta = 0$, that is, by $\beta = \dfrac{\pi}{2}$. This shows that the eccentric angles of the two extremities of a diameter differ by two right angles.

The point of bisection of the chord is—

$$x = \frac{a}{2}\left(\cos \overline{\alpha + \beta} + \cos \overline{\alpha - \beta}\right) = a \cos \alpha \cos \beta,$$

$$y = \frac{b}{2}\left(\sin \overline{\alpha + \beta} + \sin \overline{\alpha - \beta}\right) = b \sin \alpha \cos \beta.$$

The locus of the point of bisection for this system of parallel chords is therefore obtained, by the elimination of β, in the form $\frac{y}{x} = \frac{b \sin \alpha}{a \cos \alpha}$. Hence it is the diameter through the point α.

The extremities of conjugate diameters are therefore α, $\alpha + \frac{\pi}{2}$, $\alpha + \pi$, $\alpha + \frac{3\pi}{2}$. This may be stated in the form, the eccentric angles of two extremities of conjugate diameters differ by a right angle.

144. *The ellipse. Properties and theorems.*

(i) From the equation of the curve,

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2} = \frac{(a - x)(a + x)}{a^2}.$$

Hence (Fig. 77)—

$$NP^2 : A'N \,.\, NA = b^2 : a^2.$$

(ii) If the tangent at P meets the axes at T, t,

$$CN \,.\, CT = a^2,\ Cn \,.\, Ct = b^2,$$

where n is the foot of the perpendicular from P to the minor axis.

For the tangent at (x_1, y_1) is—

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$$

At T, $$y = 0,$$

hence $$\frac{xx_1}{a^2} = 1,$$

that is, $$xx_1 = a^2,$$

or $$\text{CN} \,.\, \text{CT} = a^2.$$

At t, $$x = 0,$$

hence $$\frac{yy_1}{b^2} = 1,$$

that is, $$yy_1 = b^2,$$

or $$\text{C}n \,.\, \text{C}t = b^2.$$

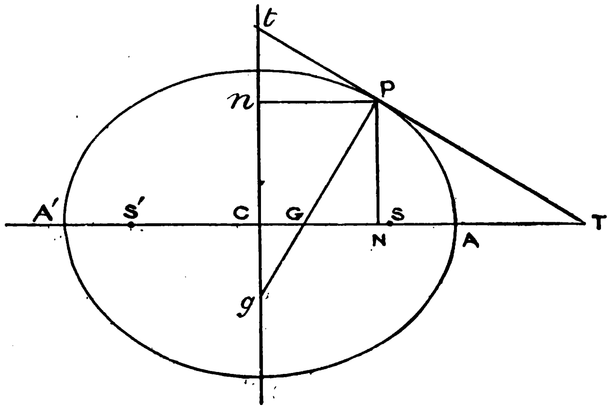

Fig. 77.

(iii) If the normal at P meets the major axis at G, then $$\text{CG} = e^2 \,.\, x_1.$$

For the normal is $$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2.$$

At G, $$y = 0,$$

hence $$\frac{a^2x}{x_1} = a^2 - b^2.$$

Since $$a^2 - b^2 = a^2e^2,$$
this becomes $$a^2x = a^2e^2x_1,$$
that is, $$x = e^2x_1.$$

Similarly $$Cg = -\frac{a^2 - b^2}{b^2}y_1.$$

Note.—For all points on the ellipse,

$$x_1 \leq a;$$
also $$e < 1,$$
and therefore $$e^2 < e,$$
hence $$e^2x_1 < ae,$$
that is, $$e^2x_1 < c.$$
This shows that $$\text{CG} < \text{CS or CS}',$$
hence that for all points on the ellipse, the normal passes between the foci.

(iv) The focal distances of (x_1, y_1) are $a - ex_1$, $a + ex_1$. To prove this, we have—

$$\text{SP}^2 = (x_1 - ae)^2 + y_1^2,$$

and $$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1.$$

Hence $$\text{SP}^2 = (x_1 - ae)^2 + b^2 - \frac{b^2x_1^2}{a^2}$$
$$= \left(1 - \frac{b^2}{a^2}\right)x_1^2 - 2aex_1 + a^2e^2 + b^2.$$

Since $$1 - \frac{b^2}{a^2} = \frac{a^2 - b^2}{a^2} = e^2, \text{ and } a^2e^2 + b^2 = a^2,$$

this is $$\text{SP}^2 = e^2x_1^2 - 2aex_1 + a^2.$$
Hence $$\text{SP} = ex_1 - a, \text{ or } a - ex_1.$$

Now whether x_1 is positive or negative, it is numerically less than a, and $e < 1$ by the definition of the ellipse Hence $ex_1 - a$ is negative, while $a - ex_1$ is positive. The correct value for SP is therefore $a - ex_1$.

$$\begin{aligned}\text{Similarly } \mathrm{S'P}^2 &= (x_1 + ae)^2 + y_1^2 \\ &= (x_1 + ae)^2 + b^2 - \frac{b^2 x_1^2}{a^2} \\ &= e^2 x_1^2 + 2aex_1 + a^2,\end{aligned}$$

from which $\mathrm{S'P} = a + ex_1$, for $-(a + ex_1)$ would give a negative value for S′P.

(v) These values for the focal distances give—

$$\mathrm{SP} + \mathrm{S'P} = 2a,$$

that is, the sum of the focal distances of a point on an ellipse is equal to the major axis.

(vi) From these values it follows also that the normal bisects the angle between the focal distances, and hence that the tangent is equally inclined to the focal distances.

To prove this, use the value for CG already found.

$$\mathrm{S'G} = c + e^2 x_1 = e(a + ex_1) = e\,.\,\mathrm{S'P},$$
$$\mathrm{SG} = c - e^2 x_1 = e(a - ex_1) = e\,.\,\mathrm{SP};$$

hence $$\mathrm{S'G} : \mathrm{GS} = \mathrm{S'P} : \mathrm{SP},$$

which proves that the normal, PG, bisects the angle SPS′.

Since the normal is the interior bisector of the angle SPS′, the tangent is the exterior bisector of this same angle. Hence the two foci of an ellipse lie on the same side of every tangent, and this is the side on which the centre lies.

(vii) The locus of the foot of the perpendicular drawn from a focus to a tangent is the auxiliary circle.

Any tangent is $y - mx = \pm\sqrt{a^2 m^2 + b^2}$;
the line through the focus perpendicular to this is—

$$my + x = \sqrt{a^2 - b^2}.$$

To find the locus of the point common to these two straight lines, eliminate m from the two equations.

Square and add the two; then—

$$y^2(1 + m^2) + x^2(m^2 + 1) = a^2(m^2 + 1),$$

that is,
$$x^2 + y^2 = a^2.$$

Notice the particular artifice by which the elimination is effected.

(viii) The product of the distances from the two foci to any tangent has the constant value b^2.

The distance from (x', y') to the tangent

$$mx - y + \sqrt{a^2m^2 + b^2} = 0,$$

written so as to be positive on the origin side of the line, is $\dfrac{mx' - y' + \sqrt{a^2m^2 + b^2}}{\sqrt{m^2 + 1}}$. Hence the product of the distances from the two foci, that is, from the points $(c, 0)$, $(-c, 0)$ is—

$$\frac{(mc + \sqrt{a^2m^2 + b^2})(-mc + \sqrt{a^2m^2 + b^2})}{m^2 + 1},$$

that is, $\dfrac{a^2m^2 + b^2 - m^2c^2}{m^2 + 1}$.

Since $a^2 - c^2 = b^2$, this is $\dfrac{m^2b^2 + b^2}{m^2 + 1}$, that is, b^2.

(ix) The perpendicular from the focus to any chord meets the directrix on the diameter that bisects the chord.

The chord $y = mx + n$ is bisected by the diameter $y = -\dfrac{b^2}{a^2m}x$. The line through S $(c, 0)$ perpendicular to

this chord is $y = -\frac{1}{m}(x - c)$. This meets the diameter where—

$$-\frac{b^2}{a^2m}x = -\frac{1}{m}(x - c),$$

that is, where $$\frac{b^2}{a^2}x = x - c;$$

hence $$x\left(1 - \frac{b^2}{a^2}\right) = c,$$

$$e^2x = ae,$$

$$x = \frac{a}{e};$$

the point is therefore on the directrix.

(x) The semi-latus rectum is a harmonic mean between the two segments of any focal chord.

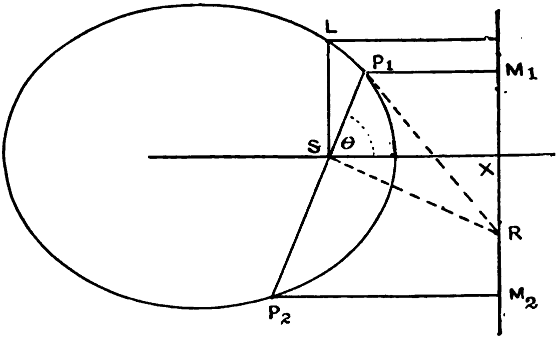

Fig. 78.

By the definition of the ellipse,

$$\begin{aligned} SP_1 &= e\,.\,P_1M_1 \text{ (Fig. 78)} \\ &= e(SX - SP_1 \cos\theta). \end{aligned}$$

Hence $$\mathrm{SP}_1(1 + e\cos\theta) = e\,.\,\mathrm{SX} = \mathrm{SL}.$$

Similarly $$\mathrm{SP}_2(1 - e\cos\theta) = \mathrm{SL}.$$

From these, $$\frac{\mathrm{SL}}{\mathrm{SP}_1} + \frac{\mathrm{SL}}{\mathrm{SP}_2} = 1 + e\cos\theta + 1 - e\cos\theta = 2,$$

that is, $$\frac{1}{\mathrm{SP}_1} + \frac{1}{\mathrm{SP}_2} = \frac{2}{\mathrm{SL}},$$

the desired result, since SL is the semi-latus rectum.

(xi) If the tangent at P meets the directrix at R, then SR is perpendicular to SP (Fig. 78).

The tangent at (x_1, y_1) is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$;

the directrix is $$x = \frac{a}{e}.$$

Hence the point R is $\left(\frac{a}{e}, \frac{b^2(ae - x_1)}{aey_1}\right)$, and the slope of SR is—

$$\frac{b^2(ae - x_1)}{aey_1} \div \left(\frac{a}{e} - ae\right),$$
$$= \frac{b^2(ae - x_1)}{a^2(1 - e^2)y_1}$$
$$= \frac{ae - x_1}{y_1},$$

since $a^2(1 - e^2) = b^2$.

Again, the slope of SP is $\frac{y_1}{x_1 - ae}$, and the product of these two slopes is -1, which proves the property stated.

Alternative statement.—The part of the tangent included between the point of contact and the directrix subtends a right angle at the focus.

Note.—The tangents at the extremities of a focal chord intersect on the directrix, at R, as in the case of the parabola, but they are not at right angles. See the next property.

(xii) The locus of the point of intersection of perpendicular tangents is the circle $x^2 + y^2 = a^2 + b^2$.

Since the slopes of perpendicular lines are m and $-\dfrac{1}{m}$, the point of intersection of perpendicular tangents is given by the equations—

$$y - mx = \sqrt{a^2m^2 + b^2},$$
$$my + x = \sqrt{a^2 + b^2m^2}.$$

The locus of the point is obtained by eliminating m. Square and add; then—

$$y^2(1 + m^2) + x^2(m^2 + 1) = a^2(m^2 + 1) + b^2(1 + m^2),$$

hence $$x^2 + y^2 = a^2 + b^2.$$

(xiii) It has been shown that a symmetrical relation connects diameters; they fall into pairs of "conjugate diameters" with the property that each bisects chords parallel to the other. The lengths of conjugate semi-diameters CP, CD are connected by the relation—

$$CP^2 + CD^2 = a^2 + b^2 \text{ (§ 118)}.$$

(xiv) The length of CD can be expressed in various ways in terms of the coordinates of P.

$$CD^2 = x_2{}^2 + y_2{}^2$$
$$= \frac{a^2y_1{}^2}{b^2} + \frac{b^2x_1{}^2}{a^2} = a^2b^2\left(\frac{x_1{}^2}{a^4} + \frac{y_1{}^2}{b^4}\right).$$

Also $$CD^2 = a^2\left(1 - \frac{x_1^{\,2}}{a^2}\right) + \frac{b^2x_1^{\,2}}{a^2}$$
$$= a^2 - \frac{a^2 - b^2}{a^2}x_1^{\,2} = a^2 - e^2x_1^{\,2}.$$

From this last form there follows at once the relation $SP \cdot S'P = CD^2$. For $SP = a - ex_1$, and $S'P = a + ex_1$ (iv).

(xv) Since the tangents at P, P′ are parallel to CD, and

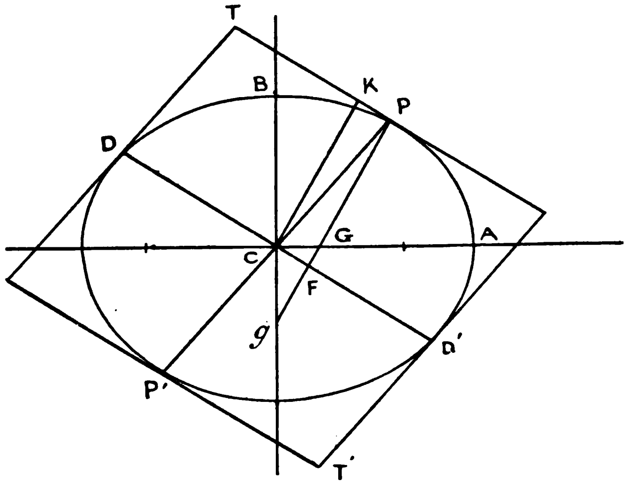

Fig. 79.

the tangents at D, D′ are parallel to CP, the tangents at the extremities of conjugate diameters form a parallelogram, whose sides are equal to the diameters. Further, this parallelogram is of constant area $4ab$. For if CK be the distance from C to the tangent at P, the area of the parallelogram CPTD (Fig. 79) is $CK \cdot CD$, where—

$$\mathrm{CK} = \frac{1}{\sqrt{\dfrac{x_1^{\,2}}{a^4} + \dfrac{y_1^{\,2}}{b^4}}}, \text{ and } \mathrm{CD} = ab\sqrt{\frac{x_1^{\,2}}{a^4} + \frac{y_1^{\,2}}{b^4}}.$$

Hence $\quad \mathrm{CK}\,.\,\mathrm{CD} = ab,$

that is, the area of the whole parallelogram is $4ab$.

(xvi) If the normal at P meets the axes at G, g, and the diameter conjugate to CP at F, then $\mathrm{PF}\,.\,\mathrm{PG} = b^2$, $\mathrm{PF}\,.\,\mathrm{P}g = a^2$ (Fig. 79).

The point G is $(e^2x_1, 0)$; P is (x_1, y_1). Hence—

$$\begin{aligned}\mathrm{PG}^2 &= [x_1(1 - e^2)]^2 + y_1^{\,2} \\ &= \left(\frac{b^2x_1}{a^2}\right)^2 + y_1^{\,2} \\ &= b^4\left(\frac{x_1^{\,2}}{a^4} + \frac{y_1^{\,2}}{b^4}\right);\end{aligned}$$

and

$$\mathrm{PF}^2 = \mathrm{CK}^2 = \frac{1}{\dfrac{x_1^{\,2}}{a^4} + \dfrac{y_1^{\,2}}{b^4}}.$$

Hence $\mathrm{PG}^2\,.\,\mathrm{PF}^2 = b^4$, $\mathrm{PG}\,.\,\mathrm{PF} = b^2$. By a similar proof, $\mathrm{P}g\,.\,\mathrm{PF} = a^2$.

Note.—These relations may be expressed in the form—

$$\mathrm{PG} : \mathrm{CD} = b : a, \quad \mathrm{P}g : \mathrm{CD} = a : b.$$

(xvii) The locus of the point of intersection of normals at consecutive (indistinguishable) points is—

$$\left(\frac{ax}{c^2}\right)^{\frac{2}{3}} + \left(\frac{by}{c^2}\right)^{\frac{2}{3}} = 1.$$

This is most easily proved by means of the eccentric angle. The normals at the points $\alpha \pm \beta$ are—

$$\frac{ax}{\cos(\alpha \pm \beta)} - \frac{by}{\sin(\alpha \pm \beta)} = c^2,$$

that is—

$$ax \sin(\alpha + \beta) - by \cos(\alpha + \beta) = c^2 \sin \overline{\alpha + \beta} \cos \overline{\alpha + \beta}$$
$$= \frac{c^2}{2} \sin(2\alpha + 2\beta),$$

and

$$ax \sin(\alpha - \beta) - by \cos(\alpha - \beta) = \frac{c^2}{2} \sin(2\alpha - 2\beta).$$

Hence, by addition and subtraction,

$$ax \sin\alpha \cos\beta - by \cos\alpha \cos\beta = \frac{c^2}{2} \sin 2\alpha \cos 2\beta,$$

$$ax \cos\alpha \sin\beta + by \sin\alpha \sin\beta = \frac{c^2}{2} \cos 2\alpha \sin 2\beta.$$

Since $\sin 2\beta = 2 \sin\beta \cos\beta$, the second of these is—

$$ax \cos\alpha + by \sin\alpha = \frac{c^2}{2} \cdot 2 \cos 2\alpha \cos\beta.$$

No lack of definiteness is caused by making the two normals become indistinguishable; write therefore $\beta = 0$; the point of intersection of the normal at α with the consecutive normal is given by—

$$ax \sin\alpha - by \cos\alpha = \frac{c^2}{2} \sin 2\alpha,$$

$$ax \cos\alpha + by \sin\alpha = \frac{c^2}{2} . 2 \cos 2\alpha.$$

To obtain the locus of (x, y), eliminate α. The best plan in this case is to solve for x, y. We find at once—

$$ax = \frac{c^2}{2}(2 \cos 2\alpha \cos\alpha + \sin 2\alpha \sin\alpha),$$

$$by = \frac{c^2}{2}(2 \cos 2\alpha \sin\alpha - \sin 2\alpha \cos\alpha).$$

Since $\cos 2\alpha \cos \alpha + \sin 2\alpha \sin \alpha = \cos (2\alpha - \alpha) = \cos \alpha,$
and $\cos 2\alpha \sin \alpha - \sin 2\alpha \cos \alpha = - \sin (2\alpha - \alpha) = - \sin \alpha,$
these give—

$$ax = \frac{c^2}{2} (\cos 2\alpha \cos \alpha + \cos \alpha),$$

$$by = \frac{c^2}{2} (\cos 2\alpha \sin \alpha - \sin \alpha),$$

that is,
$$ax = \frac{c^2}{2} \cos \alpha (\cos 2\alpha + 1),$$

$$by = \frac{c^2}{2} \sin \alpha (\cos 2\alpha - 1);$$

hence
$$ax = \frac{c^2}{2} \cos \alpha \times - 2 \cos^2 \alpha,$$

$$by = \frac{c^2}{2} \sin \alpha \times - 2 \sin^2 \alpha,$$

that is,
$$x = \frac{c^2}{a} \cos^3 \alpha, \ y = - \frac{c^2}{b} \sin^3 \alpha.$$

From these—

$$\cos \alpha = \left(\frac{ax}{c^2}\right)^{\frac{1}{3}}, \sin \alpha = \left(- \frac{by}{c^2}\right)^{\frac{1}{3}} = - \left(\frac{by}{c^2}\right)^{\frac{1}{3}},$$

and these values, substituted in the identity

$$\cos^2 \alpha + \sin^2 \alpha = 1,$$

give the equation of the evolute in the irrational form—

$$\left(\frac{ax}{c^2}\right)^{\frac{2}{3}} + \left(\frac{by}{c^2}\right)^{\frac{2}{3}} = 1.$$

Note.—Since the normal at ϕ is $- \frac{ax}{\cos \phi} + \frac{by}{\sin \phi} + c^2 = 0$, the line $\xi x + \eta y + 1 = 0$ is a normal if

$$\xi = - \frac{a}{c^2 \cos \phi}, \ \eta = \frac{b}{c^2 \sin \phi},$$

that is, if
$$c^2 \cos \phi = - \frac{a}{\xi}, \ c^2 \sin \phi = \frac{b}{\eta}.$$

Hence ξ, η must satisfy the relation (obtained from $\cos^2\phi + \sin^2\phi = 1$),

$$\frac{a^2}{\xi^2} + \frac{b^2}{\eta^2} = c^4,$$

or

$$c^4\xi^2\eta^2 - b^2\xi^2 - a^2\eta^2 = 0.$$

This is the line-equation of the evolute, which is therefore a curve of the fourth class.

The point-equation when rationalised will be found to be of the sixth degree; hence the evolute is of the sixth order. To rationalise $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = c^{\frac{4}{3}}$, cube, re-arrange, and cube again, thus obtaining—

$$(ax)^2 + by^2 + 3(ax)^{\frac{2}{3}}(by)^{\frac{2}{3}}[(ax)^{\frac{2}{3}} + (by^{\frac{2}{3}})] = c^4,$$

that is,

$$(ax)^2 + (by)^2 - c^4 = -3(ax)^{\frac{2}{3}}(by)^{\frac{2}{3}}c^{\frac{4}{3}},$$

and finally,

$$[(ax)^2 + (by)^2 - c^4]^3 + 27(ax)^2(by)^2c^4 = 0.$$

(xviii) Since the evolute is of the fourth class, there are four tangents to it from a point (h, k), that is, there are four normals to an ellipse. This can be proved independently.

First method.—Since $y = mx + \sqrt{a^2m^2 + b^2}$ is a tangent at $\left(\frac{-a^2m}{\sqrt{a^2m^2 + b^2}}, \frac{b^2}{\sqrt{a^2m^2 + b^2}}\right)$, the normal is—

$$x + \frac{a^2m}{\sqrt{a^2m^2 + b^2}} + m\left(y - \frac{b^2}{\sqrt{a^2m^2 + b^2}}\right) = 0.$$

This is to pass through the given point (h, k); hence m must satisfy—

$$h + \frac{a^2m}{\sqrt{a^2m^2 + b^2}} + mk - \frac{b^2m}{\sqrt{a^2m^2 + b^2}} = 0,$$

that is,

$$(mk + h)\sqrt{a^2m^2 + b^2} + m(a^2 - b^2) = 0.$$

This equation for m, rationalised, becomes—

$$(mk + h)^2(a^2m^2 + b^2) = m^2(a^2 - b^2)^2,$$

which is of the fourth degree. Hence there are four values for m, and therefore four normals from (h, k).

Second method.—The normal at (x_1, y_1) is—

$$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2.$$

This passes through h, k if $\dfrac{a^2h}{x_1} - \dfrac{b^2k}{y_1} = a^2 - b^2.$

Hence the normal at (x_1, y_1) passes through (h, k) if (x_1, y_1) is any point common to the ellipse and the curve $\dfrac{a^2h}{x} - \dfrac{b^2k}{y} = a^2 - b^2$, that is—

$$(a^2 - b^2)xy + b^2kx - a^2hy = 0.$$

This meets $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ in four points, whose abscissæ are given by the equation of the fourth degree which we obtain by eliminating y.

The curve $(a^2 - b^2)xy + b^2kx - a^2hy = 0$ is a rectangular hyperbola, with its asymptotes parallel to the axes of the ellipse (§ 138); it passes through the centre of the ellipse, and through the given point (h, k).

Examples on the Ellipse.

1. Find the locus of the point of intersection of tangents at the extremities of conjugate diameters.

2. Show that the locus of the point of intersection of normals at the extremities of conjugate diameters is—

$$2(a^2x^2 + b^2y^2)^3 = c^4(a^2x^2 - b^2y^2)^2.$$

3. Find the locus of the point of bisection of the chord that joins the extremities of conjugate diameters.

4. Find the envelope of the chord that joins the extremities of

conjugate diameters. Show that it is an ellipse, whose axes are one half the axes of the original ellipse, and that it is the same as the ellipse found (Ex. 3) as the locus of the point of bisection of the chord.

5. The perpendicular from a focus to the tangent at an extremity of the latus rectum through the other focus meets the tangent on the minor axis.

6. The normal at P meets the major axis at G; prove that if GP is divided in any given ratio at Q, the locus of Q is an ellipse.

7. Two points p, q on the auxiliary circle subtend a constant angle at the centre. If P, Q are the corresponding points on the ellipse, find (i) the locus of the intersections of tangents at P, Q, (ii) the locus of the point of bisection of the chord PQ, (iii) the envelope of the chord PQ.

8. Prove that the sum of the squares of the reciprocals of perpendicular diameters is constant.

9. Prove that the chord that joins the extremities of perpendicular diameters is at a constant distance from the centre.

10. Find, for the chord that joins the extremities of perpendicular diameters, (i) the locus of the pole, (ii) the locus of the point of bisection, (iii) the envelope.

11. Prove that the locus of the intersection of normals to the ellipse and the auxiliary circle at corresponding points is a circle.

12. Find the locus of the intersection of normals to two ellipses, with the same major axis, at points that lie on a line perpendicular to the major axis.

13. Find the locus of the foot of the perpendicular from the centre to a tangent.

14. Find the locus of the foot of the perpendicular from the centre to a chord that joins the extremities of conjugate diameters.

15. Find the locus of the point of bisection of the chord that joins the points of contact of perpendicular tangents.

16. Find the envelope of the chord of contact of perpendicular tangents.

17. The circle on SS′ as diameter meets the minor axis at Σ, Σ'. Prove that the sum of the squares of the distances from Σ, Σ' to any tangent to the ellipse is constant.

18. A line of constant length moves with its extremities on the axes of coordinates. Prove that the different points of the line describe ellipses.

19. Find the locus of the foot of the perpendicular from a vertex to a tangent.

20. The tangent at a point meets the tangents at the extremities of the major axis at points K, K′. Prove that KK′ subtends a right angle at either focus.

21. Prove that AK . A′K′ $= b^2$ (Ex. 20).

22. Prove that the locus of the point of intersection of SK and S′K′ (Ex. 20) is an ellipse through the points S, S′. Find the foci of this ellipse. Find also the eccentricity in terms of the eccentricity of the given ellipse.

23. Find the eccentric angles of the extremities of each latus rectum.

24. Find the point of intersection of consecutive normals at each of the four vertices.

25. Show that the point of intersection of consecutive normals at the vertex A lies between the centre and the focus S, and that the point of intersection of consecutive normals at the vertex B lies beyond the centre, but that it may be on either side of the vertex B′.

26. Find the eccentricity, if consecutive normals at one extremity of the minor axis intersect at the other extremity of the minor axis.

27. Show that the normal at an extremity of a latus rectum passes through an extremity of the minor axis if $e^4 + e^2 = 1$.

28. If the normals at P, Q are perpendicular, they meet on the diameter that bisects the chord PQ.

29. Find the locus of the pole of a tangent to $x^2 + y^2 = k^2$ with respect to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

30. Find the locus of the point of intersection of tangents that are inclined at a constant angle.

31. The lines PS, PS′ meet the diameter conjugate to CP at E, E′. Prove that PE $=$ PE′ $= a$.

32. Show that the coordinates of the point ϕ on an ellipse can be expressed as $\dfrac{a(1 - \lambda^2)}{1 + \lambda^2}$, $\dfrac{2b\lambda}{1 + \lambda^2}$, where $\lambda = \tan \dfrac{\phi}{2}$. Find the equation of the tangent at the point λ. Find the coordinates of the point of intersection of the tangents at λ_1, λ_2.

33. What relation is satisfied by the parameters λ_1, λ_2 (Ex. 32) of the opposite extremities of a diameter? Also by the parameters of the extremities of conjugate diameters?

34. The perpendicular bisector of a chord passes through a fixed point (h, k); find the locus of the point of bisection of the chord.

35. The normal at a point P passes through the vertex B′. Find the coordinates of P.

36. In which quadrant does the normal at (x_1, y_1) meet the curve again?

III

145. *The hyperbola. Formulæ and equations.*—The formulæ given for the ellipse and the properties deduced from these hold, with slight changes, for the hyperbola. Hence the proofs are not repeated.

Definition.—A hyperbola is the locus of a point whose distance from a fixed point, the focus, is in a constant ratio, greater than unity, to its distance from a fixed line, the directrix. The constant ratio is called the eccentricity.

Equation.—The simplest form of the equation is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, obtained by taking for the origin the point that divides the distance from the focus to the directrix externally in the ratio $e^2:1$, where e is the eccentricity (§ 71). There are two foci $(\pm ae, 0)$; and two corresponding directrices $x = \pm \frac{a}{e}$. For ae, c is written; $c^2 = a^2 + b^2$, $e = \frac{\sqrt{a^2 + b^2}}{a}$.

The length of the latus rectum (§ 73) is $2\frac{b^2}{a}$; it lies on the line $x = ae$.

The line-equation of the hyperbola is $a^2\xi^2 - b^2\eta^2 = 1$.

Equation of tangent with a given slope.—The tangent

with slope m is $y = mx \pm \sqrt{a^2m^2 - b^2}$; the point of contact is $\left(\mp \frac{a^2m}{\sqrt{a^2m^2 - b^2}}, \frac{\mp b^2}{\sqrt{a^2m^2 - b^2}}\right)$ (§ 79).

Equations of tangent and normal at a given point.—The tangent at (x_1, y_1) is—

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1 \text{ (§ 91)};$$

the normal at (x_1, y_1) is—

$$\frac{a^2}{y_1}(x - x_1) + \frac{b^2}{y_1}(y - y_1) = 0,$$

that is,
$$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 + b^2.$$

Equation of chord.—The chord through (x_1, y_1), (x_2, y_2) is $\frac{x(x_1 + x_2)}{a^2} - \frac{y(y_1 + y_2)}{b^2} = \frac{x_1x_2}{a^2} - \frac{y_1y_2}{b^2} + 1$; the slope of this chord is $\frac{b^2(x_1 + x_2)}{a^2(y_1 + y_2)}$ (§ 91).

Pole and polar.—The polar of (x', y') is $\frac{xx'}{a^2} - \frac{yy'}{b^2} = 1$ (§ 97).

Equation of diameter.—Chords of slope m_1 are bisected by the diameter $y = m_2x$, where $m_1m_2 = \frac{b^2}{a^2}$ (§ 108).

If a semi-diameter CP of length r makes with the axis of x an angle θ, the coordinates of P are $r \cos\theta$, $r \sin\theta$ (§ 22). Hence—

$$\frac{r^2 \cos^2\theta}{a^2} - \frac{r^2 \sin^2\theta}{b^2} = 1,$$

that is,
$$r^2\left(\frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2}\right) = 1,$$

which may be written in the form—

$$r^2(e^2 \cos^2\theta - 1) = b^2.$$

Hence the length of the diameter is real if $\cos^2\theta > \frac{1}{e^2}$, that is, if $\tan^2\theta < \frac{b^2}{a^2}$; it is imaginary if $\tan^2\theta > \frac{b^2}{a^2}$. If $e^2\cos^2\theta = 1$, that is, if $\tan^2\theta = \frac{b^2}{a^2}$, the length of the diameter becomes infinite. (Compare § 124.)

Asymptotes.—The asymptotes are $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$; the curve lies in one pair of angles formed by these, and continually approaches the asymptotes towards infinity (§ 127).

Conjugate hyperbola.—The hyperbola conjugate to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$. It has the same asymptotes, and the same directions for conjugate diameters. On one of a pair of conjugate diameters one hyperbola determines a real length; on the other diameter of the pair a real length is determined by the other hyperbola.

If (x_1, y_1) is a real point P on one hyperbola, and (x_2, y_2) a point in which the diameter conjugate to CP meets the conjugate hyperbola, then $\frac{y_2}{b} = \pm\frac{x_1}{a}$, and $\frac{x_2}{a} = \pm\frac{y_1}{b}$ (§ 132).

The line Pd that joins corresponding points on the two hyperbolas (Fig. 81) is parallel to one asymptote and bisected by the other.

The auxiliary circle.—For the hyperbola the circle on the major axis, that is, the auxiliary circle, is not used to the same purpose as for the ellipse. It does not lead to a convenient expression for the coordinates of a point.

146. *The hyperbola. Properties and theorems.*

(i) $NP^2 : A'N . AN = b^2 : a^2$ (Fig. 80).

(ii) $CN . CT = a^2$; $Cn . Ct = -b^2$.

(iii) $CG = e^2 . x_1$; $Cg = \dfrac{a^2 + b^2}{b^2} y_1$.

For all points on the right-hand branch of the hyperbola,

$$x_1 \geq a;$$

also $$e > 1,$$

and therefore $$e^2 > e,$$

hence $$e^2 x_1 > ae,$$

that is, $$e^2 x_1 > c.$$

This shows that $CG > CS$; the normal does not pass between the foci.

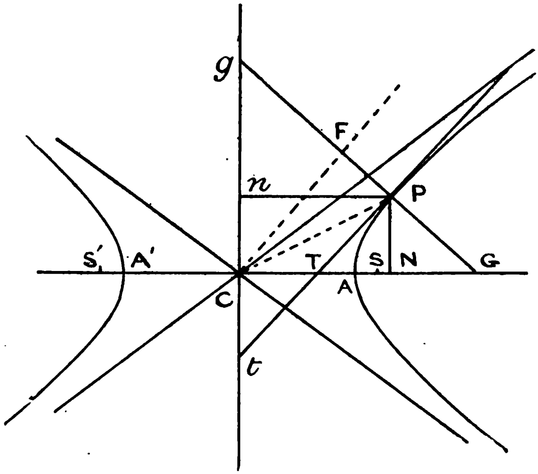

Fig. 80.

(iv) The focal distances of a point P on the right-hand

branch are $SP = ex_1 - a$, $S'P = ex_1 + a$; for a point P' on the left-hand branch they are $SP' = a - ex_1$, $S'P' = -a - ex_1$. (By the same proof as for the ellipse, with due attention to sign.)

(v) Hence for a point on the right-hand branch, $S'P - SP = 2a$, and for a point on the left-hand branch, $SP' - S'P = 2a$; that is, the difference of the focal distances of a point on the hyperbola is equal to the major axis.

(vi) The normal is equally inclined to the focal distances; the tangent bisects the angle between the focal distances.

The foci are on the same side of any normal, but on opposite sides of any tangent.

(vii) The locus of the foot of the perpendicular drawn from a focus to a tangent is the auxiliary circle.

(viii) The product of the distances from the two foci to any tangent has the constant value $-b^2$.

(ix) The perpendicular from the focus to any chord meets the directrix on the diameter that bisects the chord.

(x) The semi-latus rectum is a harmonic mean between the two segments of any focal chord.

(xi) If the tangent at P meets the directrix at R, then SR is perpendicular to SP.

(xii) The locus of the point of intersection of perpendicular tangents is the circle $x^2 + y^2 = a^2 - b^2$. Hence a hyperbola has no real tangents at right angles if $a^2 < b^2$; and if $a^2 = b^2$, the asymptotes are the only tangents at right angles.

(xiii) The lengths of conjugate semi-diameters of conjugate hyperbolas are connected by the relation

$CP^2 - Cd^2 = a^2 - b^2$ (§ 135). Hence, as a particular case, conjugate semi-diameters of conjugate rectangular hyperbolas are equal.

(xiv) If P is (x_1, y_1), d is $\left(\frac{ay_1}{b}, \frac{bx_1}{a}\right)$; hence—

$$Cd^2 = \frac{a^2y_1^2}{b^2} + \frac{b^2x_1^2}{a^2} = a^2b^2\left(\frac{x^2}{a^4} + \frac{y_1^2}{b^4}\right).$$

Also $$Cd^2 = a^2\left(\frac{x_1^2}{a^2} - 1\right) + \frac{b^2x_1^2}{a^2}$$

$$= \frac{a^2 + b^2}{a^2}x_1^2 - a^2 = e^2x_1^2 - a^2.$$

Hence $$Cd^2 = SP . S'P \text{ (iv)}.$$

(xv) Tangents to conjugate hyperbolas at correspond-

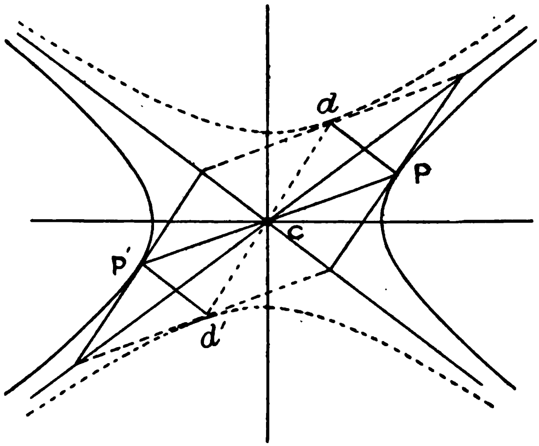

Fig. 81.

ing points form a parallelogram, of constant area $4ab$ (Fig. 81). The asymptotes are the diagonals of this

parallelogram; for the line joining the origin to the point of intersection of

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1, \quad \frac{xx_2}{a^2} - \frac{yy_2}{b^2} = -1,$$

is
$$\frac{x(x_1 + x_2)}{a^2} - \frac{y(y_1 + y_2)}{b^2} = 0.$$

Now $\frac{x_1 + x_2}{a} = \frac{x_1}{a} + \frac{y_1}{b}$, and $\frac{y_1 + y_2}{b} = \frac{y_1}{b} + \frac{x_1}{a}$, hence this line is $\frac{x}{a} - \frac{y}{b} = 0$, an asymptote.

(xvi) $\mathrm{PF}.\mathrm{PG} = -b^2$, $\mathrm{PF}.\mathrm{P}g = a^2$ (Fig. 80).

(xvii) The point-equation of the locus of the point of intersection of normals at consecutive points, that is, the evolute of the hyperbola, is—

$$\left(\frac{ax}{c^2}\right)^{\frac{2}{3}} - \left(\frac{by}{c^2}\right)^{\frac{2}{3}} = 1,$$

or, rationalised—

$$[(ax)^2 - (by)^2 - c^4]^3 - 27(ax)^2(by)^2c^4 = 0.$$

The line-equation is $c^4\xi^2\eta^2 + b^2\xi^2 - a^2\eta^2 = 0$.

The proof for the line-equation is the same as in the case of the ellipse. The point-equation is deduced from that for the ellipse by changing the sign of b^2; but the proof given, depending on the eccentric angle, is not available for the hyperbola. The proof is given later (§ 193).

(xviii) There are four normals from a point (h, k) to a hyperbola. The points at which they must be drawn are determined by the rectangular hyperbola—

$$(a^2 + b^2)xy - b^2kx - a^2hy = 0.$$

(xix) The segments of any line included between the hyperbola and its asymptotes are equal (§ 128).

(xx) The part of a tangent to a hyperbola that is included between the asymptotes is bisected at the point of contact (§ 129).

Examples on the Hyperbola.

1. The triangle included between the asymptotes and any tangent is of constant area.

2. The asymptotes meet the directrices on the auxiliary circle.

3. The part of the normal at P to a rectangular hyperbola that is included between the axes of the curve is bisected at P. Also this part of the normal is equal to the part of the tangent that is included between the asymptotes.

4. Show that the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$, and the hyperbola $\frac{x^2}{12}-\frac{y^2}{4}=1$, have the same foci, and that the tangents to the two curves at any of their four points of intersection are at right angles.

5, 6. See Nos. 5, 6 for the ellipse (p. 271).

7. Find the locus of a point whose polars with respect to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are perpendicular.

8, 9, 10. See Nos. 8, 9, 10 for the ellipse (p. 271).

11. Prove that the polars of any point with respect to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ make supplementary angles with the axis of x, and meet this axis at the same point.

12. Two secants are drawn from a point on the axis of x, to make supplementary angles with the axis. Prove that the two tangents to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the points where it is met by one secant, and the two tangents to $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the points where it is met by the other secant, all pass through one point.

13, 14. See Nos. 13, 14 for the ellipse (p. 271).

15. Two hyperbolas are drawn with the same asymptotes. Show that the two intercepts between the two curves on any line are equal.

16. The asymptotes of one rectangular hyperbola are the axes of a second rectangular hyperbola. Prove that the curves have precisely two real common points; and that at a common point the tangents to the two curves are at right angles.

Note.—When the tangents to two curves at a common point are at right angles, the curves are said to cut at right angles, or orthogonally; they are also said to be orthogonal.

17. Prove that all curves of the system $x^2 - y^2 = \lambda$ (obtained by varying λ) are orthogonal to all curves of the system $2xy = \mu$ (obtained by varying μ).

18. Prove that the sum of the squares of the distances from the foci of a hyperbola to any tangent to the conjugate hyperbola is constant.

19. See No. 19 for the ellipse (p. 272).

20, 21, 22. Examine Nos. 20, 21, 22 for the ellipse (p. 272), and determine whether, and in what form, they hold for the hyperbola.

23. See No. 31 for the ellipse (p. 272).

24. See No. 34 for the ellipse (p. 273).

25. Prove that through any given point there can be drawn two conics with given foci, an ellipse and a hyperbola, and that these are orthogonal.

26. Find where the normal at (x_1, y_1) meets the curve again.

27. Find the points at which the normal is parallel to an asymptote.

28. Find for what part of the curve the normals meet the other branch.

29. Show that the coordinates of a point on a hyperbola can be expressed as $a\dfrac{(1+\lambda^2)}{1-\lambda^2}, \dfrac{2b\lambda}{1-\lambda^2}$. Find the equation of the tangent and normal at the point λ, and the coordinates of the point of intersection of the tangents at λ_1, λ_2.

30. What relation is satisfied by the parameters λ_1, λ_2 (Ex. 29) of the opposite extremities of a diameter?

31. Find in terms of λ the coordinates of the points on the conjugate hyperbola that correspond to the point λ on the given hyperbola.

32. The circle described with its centre at a point P on a rectangular hyperbola, to pass through the centre of the hyperbola, cuts the asymptotes on the tangent at P, and the axes on the normal at P.

IV

147. *The circle.*—From the point of view of conics, the circle can be looked upon as a particular case of the ellipse. The equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, when $a = b$, represents a circle whose centre is at the origin.

So far as the circle itself is concerned, the methods of analytical geometry are not necessary; pure geometry affords elegant proofs and constructions. But inasmuch as circles are constantly met with in problems that demand the use of analytical geometry, it is necessary to consider circles with the help of analytical methods.

If the centre is C(p, q), and radius r, the equation of the circle (the analytical expression of the defining property CP $= r$) is—

$$(x - p)^2 + (y - q)^2 = r^2,$$

that is, $x^2 + y^2 - 2px - 2qy + p^2 + q^2 - r^2 = 0,$

or, after multiplication by any quantity a,

$$ax^2 + ay^2 + 2gx + 2fy + c = 0.$$

Thus the general equation of the second degree,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

does not represent a circle unless $a = b, h = 0$. If these conditions are satisfied the equation can be written—

$$x^2 + 2\frac{g}{a}x + y^2 + 2\frac{f}{a}y + \frac{c}{a} = 0,$$

that is, by completing the squares,

$$\left(x + \frac{g}{a}\right)^2 + \left(y + \frac{f}{a}\right)^2 = \frac{f^2 + g^2 - ac}{a^2}.$$

Since this is equivalent to $CP^2 = r^2$, where C is $\left(-\frac{g}{a}, -\frac{f}{a}\right)$, and $r = \frac{\sqrt{f^2 + g^2 - ac}}{a^2}$, it is the equation of a circle.

In general, the equation is written—

$$x^2 + y^2 + 2gx + 2fy + c = 0;$$

the centre is $(-g, -f)$, and the radius is $\sqrt{f^2 + g^2 - c}$. If the centre is O, the equation is—

$$x^2 + y^2 = r^2.$$

Equation of tangent.—The tangent with slope m to $x^2 + y^2 = r^2$ is $y = mx \pm r\sqrt{m^2 + 1}$; the point of contact is $\left(\frac{\mp rm}{\sqrt{m^2 + 1}}, \frac{\pm r}{\sqrt{m^2 + 1}}\right)$.

The tangent at (x_1, y_1) to $x^2 + y^2 = r^2$ is $xx_1 + yy_1 = r^2$; to $x^2 + y^2 + 2gx + 2fy + c = 0$, it is—

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

The normal to $x^2 + y^2 = r^2$ is $y_1(x - x_1) - x_1(y - y_1) = 0$, that is, $xy_1 - x_1y = 0$; hence the normal passes through the centre.

Equation of polar.—The polar of (x', y') with respect to $x^2 + y^2 = r^2$, or $x^2 + y^2 + 2gx + 2fy + c = 0$ is—

$$xx' + yy' = r^2, \text{or } xx' + yy' + g(x + x') + f(y + y') + c = 0.$$

The polar of a point T with respect to a circle of centre

C is perpendicular to CT, and meets CT at N, where CN . CT $= r^2$. For the equation of CT is $\frac{x}{x'} = \frac{y}{y'}$, which is perpendicular to $xx' + yy' = r^2$; and the distance from C to the line, that is, CN, is $\frac{r^2}{\sqrt{x'^2 + y'^2}}$, while CT $= \sqrt{x'^2 + y'^2}$, hence CN . CT $= r^2$.

Diameters.—A diameter is perpendicular to the chords that it bisects; for the relation $m_1 m_2 = -\frac{b^2}{a^2}$ becomes, when $b = a$, $mm' = -1$. Hence conjugate diameters are perpendicular diameters.

Asymptotes.—The asymptotes of a circle are imaginary. The line $y = mx \pm r\sqrt{m^2 + 1}$ is tangent at the point $\frac{\mp mr}{\sqrt{m^2+1}}, \frac{\pm r}{\sqrt{m^2+1}}$; this point is at infinity if $m^2 + 1 = 0$, that is, if $m = \pm i$. The asymptotes are therefore $y = \pm ix$; their combined equation is—

$$(y - ix)(y + ix) = 0,$$

that is,

$$x^2 + y^2 = 0.$$

The asymptotes of $x^2 + y^2 + 2gx + 2fy + c = 0$ are—

$$(x + g)^2 + (y + f)^2 = 0.$$

The asymptotes of all circles have the same slope; their equations are $y + f = \pm i(x + g)$. This shows that all circles pass through the same (imaginary) points at infinity.

Tangent from a point.—If P be the point of contact of a

tangent from $T(x', y')$, the right-angled triangle CPT shows that $TP^2 = CT^2 - CP^2$

$$= (x' - p)^2 + (y' - q)^2 - r^2.$$

Now the equation of the circle is $u = 0$, where

$$u = (x - p)^2 + (y - q)^2 - r^2;$$

hence $TP^2 = u'$. That is, the value of the expression u at a point, or (§ 85) the power of a point with respect to a circle, is the square of the length of the tangent from the point to the circle.

Note.—That u is positive outside the circle, negative inside, can be shown in various ways. If P be outside,

$$CP > r,$$

hence $CP^2 > r^2$,

that is, $CP^2 - r^2$ is positive,

hence $(x - p)^2 + (y - q)^2 - r^2$ is positive;

or u is positive.

Similarly for an interior point, $CP < r$, which shows that u is negative.

The power of a point with respect to a circle may also be expressed as the product of the segments of any chord through the point. This has the advantage of supplying a real geometrical interpretation for the value of u at all points of the plane (§ 42).

Power.—By the power of one circle with respect to another is meant the value of the expression $d^2 - r_1^2 - r_2^2$, where d is the distance between the centres of the two circles. If the radius of one circle is zero, so that the circle reduces to a point, the power as now defined agrees with the definition of the power of a point with respect

to a circle. If the two circles intersect at P (Fig. 82), the angle C_1PC_2, or θ, is obtained by the trigonometrical formula $C_1C_2^2 = C_1P^2 + C_2P^2 - 2C_1P \cdot C_2P \cos\theta$; hence $d^2 - r_1^2 - r_2^2 = -2r_1r_2\cos\theta$.

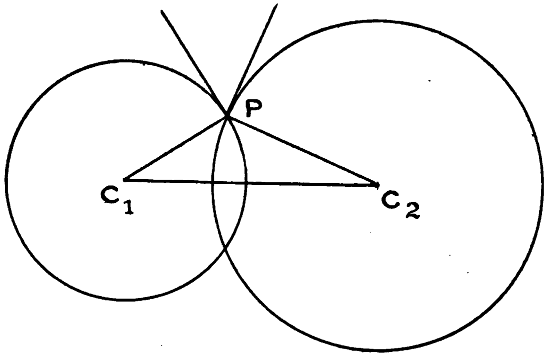

Fig. 82.

This shows that the two circles are orthogonal if their power is zero; for then $\cos\theta = 0$, hence $\theta = \frac{\pi}{2}$. If the circles touch externally, $\theta = \pi$, hence their power $= 2rr'$; if they touch internally, $\theta = 0$, and the power $= -2rr'$.

Radical axis.—The common points of two circles are obtained by solving the two equations—

$$u = x^2 + y^2 + 2gx + 2fy + c = 0,$$
$$v = x^2 + y^2 + 2g'x + 2f'y + c' = 0;$$

hence they lie on $u - v = 0$, that is, on the straight line $2(g - g')x + 2(f - f')y + c - c' = 0$. There are therefore two common points (two at a finite distance, but also two at infinity, as already shown), and the line

$u - v = 0$ joins these; it is the common chord of the circles. Even when the common points are imaginary, the line $u - v = 0$ is real. This line is called the radical axis of the two circles.

Since the centres of the circles are $(-g, -f)$, $(-g', -f')$, the equation of the line of centres is—

$$\frac{x+g}{g-g'} - \frac{y+f}{f-f'} = 0;$$

hence the radical axis is perpendicular to the line of centres.

At any point T on the radical axis, $u = v$; the interpretation of this is that the tangents from T to the two circles are equal in length.

The radical axes of three circles, $u = 0$, $v = 0$, $w = 0$, taken in pairs, meet at a point, called the radical centre of the three circles. For the three radical axes are $v - w = 0$, $w - u = 0$, $u - v = 0$; if the first two meet at a point S, then at S the values of v, w are equal, and also the values of w, u; hence the values of u, v are equal, which shows that S lies on the third radical axis. The six tangents from S to the three circles are all equal in length; a circle can be described with centre S to pass through the six points of contact, and this circle cuts the three circles $u = 0$, $v = 0$, $w = 0$ orthogonally.

Examples on the Circle.

1. Show that the equation of a circle for which (x_1, y_1), (x_2, y_2) are the extremities of a diameter can be written in the form—

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0.$$

2. Find the locus of a point which moves so that the lengths of the tangents from it to the two circles,

$$x^2 + y^2 + 6x - 8y + 1 = 0,$$
$$x^2 + y^2 + 2x + 4y - 7 = 0,$$

are in the ratio 2 : 1.

3. A point moves so that the lengths of the tangents drawn to two given circles are in a given ratio. Prove that the locus is a circle, which passes through the common points of the two given circles.

4. Show that the radical axis of two circles bisects their common tangents.

5. Find the locus of a point which moves so that the tangents from it to two equal circles differ by a constant.

6. Show that the power of the circles,

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$
$$x^2 + y^2 + 2g'x + 2f'y + c' = 0,$$

is $-2gg' - 2ff' + c + c'$.

Hence find the condition that the circles be orthogonal.

7. What is the condition that a variable circle,

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$

be orthogonal to a fixed circle $x^2 + y^2 + c' = 0$?

8. Show that if the power of a variable circle and a fixed circle is constant, the variable circle in all its positions is orthogonal to another fixed circle, concentric with the given circle.

9. Find a circle that shall be orthogonal to the three circles,

$$x^2 + y^2 + 4x - 2y = 0,$$
$$x^2 + y^2 - 6x - 4y - 14 = 0,$$
$$x^2 + y^2 + 2x + 2y + 6 = 0.$$

10. Prove that if the power of a fixed point with respect to a variable circle is constant, the circle is orthogonal to a certain fixed circle.

CHAPTER XI

CHANGE OF AXES

148. It is often necessary to change the axes of coordinates. The most general change of rectangular axes, namely, from Ox, Oy to O$'x'$, O$'y'$ (Fig. 83), can be performed by two simple changes; first change the origin, keeping the directions of the axes unchanged, that is,

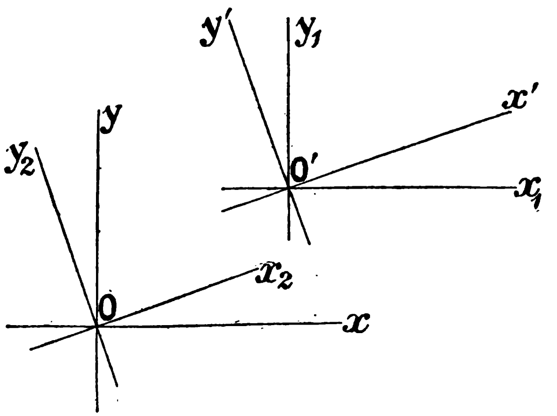

Fig. 83.

take as axes the lines O$'x_1$, O$'y_1$; and then turn the axes through a properly chosen angle, namely, the angle that O$'x'$ makes with O$'x_1$. If more convenient, begin by

turning the axes round to a position parallel to that of the desired axes (Ox_2, Oy_2), and then change the origin.

Since we wish to pass from equations that involve the original coordinates to equations in the new coordinates, we require expressions for the old x, y in terms of the new x', y'.

149. (i) *Formulæ for change of origin.*—Let the coordinates of the new origin, referred to the original axes, be p, q. Call the old coordinates of P, x, y, and the new, x', y'.

Then from the figure (Fig. 84)—

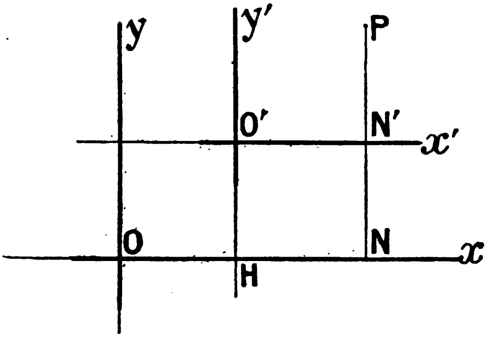

Fig. 84.

$$\begin{aligned} \text{old } x = \text{ON} &= \text{HN} + \text{OH} \\ &= \text{O}'\text{N}' + \text{OH} = x' + p \\ \text{old } y = \text{NP} &= \text{N}'\text{P} + \text{NN}' \\ &= \text{N}'\text{P} + \text{HO}' = y' + q. \end{aligned}$$

Example i.—Find what the equation $5x + 3y - 13 = 0$ becomes when the origin is changed to (2, 1).

Here $x = x' + 2$, $y = y' + 1$; hence the transformed equation is—

$$5(x' + 2) + 3(y' + 1) - 13 = 0,$$

that is,
$$5x' + 3y' = 0.$$

It is generally convenient to drop the accents after the change is completed, and use x, y for the new coordinates. The transformed equation is written $5x + 3y = 0$.

Example ii.—When the origin is changed to (7, 5), the equation

$$x^2 + y^2 - 14x - 10y - 7 = 0$$

becomes $(x + 7)^2 + (y + 5)^2 - 14(x + 7) - 10(y + 5) - 7 = 0$,

that is,
$$x^2 + y^2 = 81.$$

EXAMPLES.

1. Find what $y^2 - 8x - 8y - 24 = 0$ becomes when the origin is changed to $(-5, 4)$.

2. Find what $x^2 + 4y^2 - 6x - 40y + 100 = 0$ becomes when the origin is changed to (3, 5).

3. Find what $3x^2 + 5xy - 2y^2 - 54x + 4y - 96 = 0$ becomes when the origin is changed to (4, 6).

4. Find the coefficients of x, y when the equation

$$3x^2 + 10xy + 4y^2 + 18x + 4y - 5 = 0$$

is transformed by changing the origin to (p, q).

Find what the equation becomes when the new origin is chosen so that these two coefficients shall have the value zero.

5. What point must be taken as origin in order that the equation $5x^2 + 8xy - y^2 - 20x + 26y + 7 = 0$ shall be transformed into an equation without the terms of the first degree?

150. (ii) *Formulæ for change of direction of axes.*—The position of the new axes is determined by θ, the angle that the new axis of x, Ox' makes with the old axis of x.

Draw ON, NP, the old coordinates of P, and ON′, N′P, the new coordinates (Fig. 85). Draw N′H, N′K perpendicular to Ox, NP.

Then, from Fig. 85,

$$\begin{aligned}
\text{old } x = \text{ON} &= \text{OH} - \text{NH} = \text{OH} - \text{KN}' \\
&= \text{ON}' \cos\theta - \text{N}'\text{P} \sin\theta \\
&= x' \cos\theta - y' \sin\theta; \\
\text{old } y = \text{NP} &= \text{NK} + \text{KP} = \text{HN}' + \text{KP} \\
&= \text{ON}' \sin\theta + \text{N}'\text{P} \cos\theta \\
&= x' \sin\theta + y' \cos\theta.
\end{aligned}$$

Thus in order to turn the axes through an angle θ, write for the old x, y these expressions in terms of the new x, y.

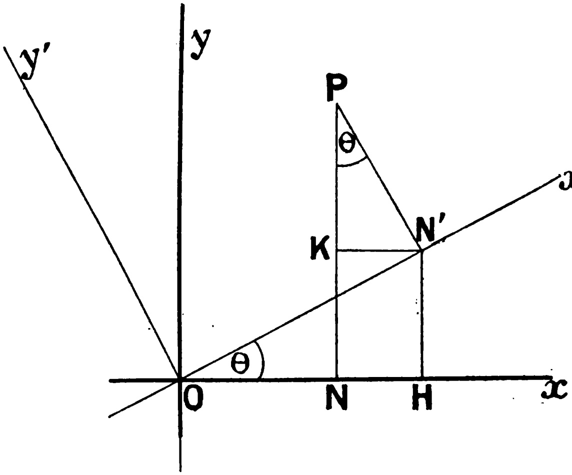

Fig. 85.

For distinctness, during the process the new coordinates may be denoted by x', y', but when the old coordinates are done with the accents become unnecessary; the letters x, y are now available for the new coordinates.

Example i.—Find what the equation $xy + 4x - 6y + 1 = 0$ becomes when the axes are turned through an angle $\frac{\pi}{4}$.

The formulæ of transformation are—

$$x = \frac{1}{\sqrt{2}}x' - \frac{1}{\sqrt{2}}y' = \frac{x' - y'}{\sqrt{2}},$$
$$y = \frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y' = \frac{x' + y'}{\sqrt{2}};$$

hence the equation becomes—

$$\frac{x'^2 - y'^2}{2} + \frac{4(x' - y')}{\sqrt{2}} - \frac{6(x' + y')}{\sqrt{2}} + 1 = 0,$$

that is, $x^2 - y^2 + 4\sqrt{2}(x - y) - 6\sqrt{2}(x + y) + 2 = 0,$

or $x^2 - y^2 - 2\sqrt{2}\,x - 10\sqrt{2}\,y + 2 = 0.$

Example ii.—Transform the equation

$$2x^2 + 24xy + 9y^2 - 6x + 2y - 1 = 0$$

by turning the axes through an angle $\tan^{-1}\left(-\frac{3}{4}\right)$.

Since $\tan\theta = -\frac{3}{4}$, the smallest positive value of θ is greater than $\frac{\pi}{2}$ and less than π; this value we will use. Then—

$$\sin\theta = +\frac{3}{5}, \cos\theta = -\frac{4}{5}.$$

Hence $x = -\frac{4}{5}x' - \frac{3}{5}y' = -\frac{4x' + 3y'}{5},$

$$y = \frac{3}{5}x' - \frac{4}{5}y' = \frac{3x' - 4y'}{5}.$$

The equation becomes—

$$2\frac{(4x + 3y)^2}{25} - 24\frac{(4x + 3y)(3x - 4y)}{25} + 9\frac{(3x - 4y)^2}{25}$$
$$+ 6\frac{4x + 3y}{5} + 2\frac{3x - 4y}{5} - 1 = 0,$$

which reduces to $-7x^2 + 18y^2 + 6x + 2y - 1 = 0.$

EXAMPLES.

1. Transform the equation $3x^2 - 3y^2 + 8x + 4y - 7 = 0$ by turning the axes through an angle $\frac{\pi}{4}$.

2. Transform the equation $8x^2 + 4xy + 5y^2 + 6x - 2y - 1 = 0$ by turning the axes through an angle $\tan^{-1}\frac{1}{2}$. Also by turning the axes through an angle $\tan^{-1}(-2)$.

3. Transform the equation $6x^2 - 4xy + 9y^2 + 2x + 4y - 3 = 0$ by turning the axes through an angle θ, chosen so as to reduce to zero the coefficient of the term xy in the transformed equation.

151. Sometimes the new axes are given by their equations. For example, it is desired to take as axes the lines $4x - y - 8 = 0$,

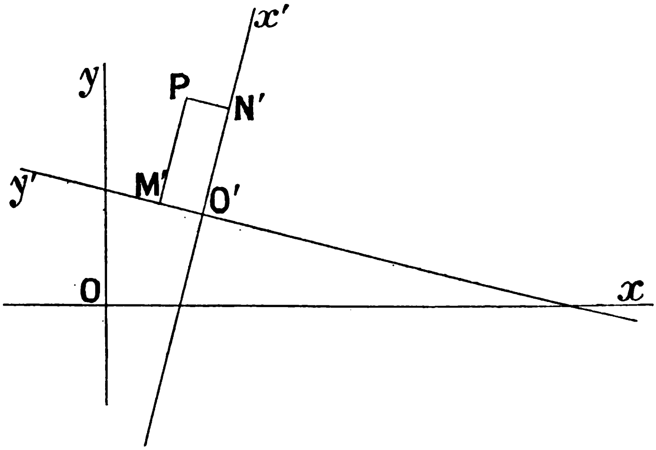

Fig. 86.

$x + 4y - 12 = 0$, the positive direction on each to be chosen so that the present origin is in the $(- \; +)$ quadrant (§ 11).

From the formula for the distance from a point to a line (§ 43),

$$x' = \text{M}'\text{P (Fig. 86)} = \frac{x + 4y - 12}{\sqrt{17}},$$

$$y' = \text{N}'\text{P} = -\frac{4x - y - 8}{\sqrt{17}},$$

in which the signs have been chosen in accordance with the specifications as to the old origin. Hence—

$$x + 4y - 12 = \sqrt{17}x',$$
$$4x - y - 8 = -\sqrt{17}\,y';$$

from which we obtain the formulæ of transformation—

$$\text{old } x = \frac{x'}{\sqrt{17}} - \frac{4y'}{\sqrt{17}} + \frac{44}{17},$$
$$\text{old } y = \frac{4x'}{\sqrt{17}} + \frac{y'}{\sqrt{17}} + \frac{40}{17}.$$

152. If the new axes are at our disposal, we may choose them so as to simplify some particular equation. For instance, if we change the origin to $(3, -1)$, $x^2 + y^2 - 6x + 2y + 1 = 0$ becomes $x^2 + y^2 = 9$; if we turn the axes through an angle $\frac{\pi}{4}$, $3x^2 + 4xy + 3y^2 = 1$ becomes $5x^2 + y^2 = 1$. The object to be gained in changing the axes is generally this reduction of an equation to a simpler form, by getting rid of some of the terms. We proceed to examine the effect produced by transformation on an equation of the second degree, in order to discover what simplification in the equation can be thereby produced. It is convenient to examine separately the effect produced—

(i) By changing the origin.
(ii) By changing the directions of the axes.

153. (i) If the new origin is (p, q), the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

becomes—

$$a(x'+p)^2 + 2h(x'+p)(y'+q) + b(y'+q)^2 + 2g(x'+p) + 2f(y'+q) + c = 0,$$

that is (after dropping the accents)—

$$ax^2 + 2hxy + by^2 + 2(ap+hq+g)x + 2(hp+bq+f)y + ap^2 + 2hpq + bq^2 + 2gp + 2fq + c = 0,$$

or $$a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c' = 0,$$

where $$a' = a,\ h' = h,\ b' = b,$$
$$g' = ap + hq + g,\ f' = hp + bq + f,$$
$$c' = ap^2 + 2hpq + bq^2 + 2gp + 2fq + c.$$

Notice that the coefficients of the terms of the second degree are unaltered by the transformation, while the other coefficients are changed, the new values depending on p, q.

If the new origin is at our disposal, it is generally convenient to choose p, q so that $g' = 0, f' = 0$; that is, p, q are determined by the equations—

$$ap + hq + g = 0,$$
$$hp + bq + f = 0.$$

Hence $$p = \frac{hf - bg}{ab - h^2},\ q = \frac{gh - af}{ab - h^2}.$$

(This choice is inadmissible, however, if $ab - h^2 = 0$, for this makes $p = \infty$, $q = \infty$; but this is the only case in which the conditions $g' = 0$, $f' = 0$ fail to give a suitable origin.) With these values for p, q, the value of c', namely,

$$ap^2 + 2hpq + bq^2 + 2gp + 2fq + c,$$

or $$p(ap + hq + g) + q(hp + bq + f) + gp + fq + c,$$

becomes $$gp + fq + c,$$

hence—

$$c' = gp + fq + c = \frac{g(hf - bg) + f(gh - af) + c(ab - h^2)}{ab - h^2}$$

$$= \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{ab - h^2}.$$

The expression in the numerator, which can be written in the form of a determinant—

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

is called the discriminant of $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$, and is denoted by D.

Hence by change of origin, the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

can be reduced to—

$$ax^2 + 2hxy + by^2 + \frac{D}{ab - h^2} = 0.$$

To reduce the equation to this form, we have taken as origin a point whose coordinates satisfy the equations—

$$ap + hq + g = 0,$$
$$hp + bq + f = 0,$$

that is, the point of intersection of the two lines—

$$ax + hy + g = 0,$$
$$hx + by + f = 0.$$

If $ab - h^2 = 0$, the two lines are parallel, and their intersection, lying at infinity, is not available for use as origin.

Note 1.—If the coefficients a, b, c, f, g, h are arranged in the scheme—

$$\begin{matrix} a & h & g \\ h & b & f \\ g & f & c \end{matrix}$$

(in which we write a, b, c in the diagonal, and then fill up with f, g, h), the coefficients in the first row are those of the first equation, $ap + hq + g = 0$; those in the second row give the second equation $hp + bq + f = 0$; and those in the third row are those needed for the new absolute term, $c' = gp + fq + c$.

Note 2.—To calculate the value of D, the easiest way is to write the coefficients in the scheme—

$$\begin{matrix} a & f & f \\ b & g & g \\ c & h & h \end{matrix}$$

writing twice the f, g, h, which occur in the expression with the numerical coefficients 2. To obtain the value of D, take the products of the coefficients in the vertical columns with positive sign, and the products of the coefficients in the horizontal rows with negative sign. This gives for D the value $abc + fgh + fgh - af^2 - bg^2 - ch^2$, which is right. But to calculate c', it is simpler not to use D; use rather $c' = gp + fq + c$, p, q having been already found.

154. It may happen that $D = 0$; the equation has then been reduced to the form $ax^2 + 2hxy + by^2 = 0$. But $ax^2 + 2hxy + by^2$ is the product of two linear factors, $\lambda x + \mu y$, $\lambda' x + \mu' y$, which are real or imaginary according as $ab - h^2$ is negative or positive. Hence the equation is $(\lambda x + \mu y)(\lambda' x + \mu' y) = 0$, which represents two lines through the new origin. That is, if

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

and $ab - h^2 \neq 0$, the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represents two straight lines, which meet at the point common to $ax + hy + g = 0$, $hx + by + f = 0$.

Even if $ab - h^2 = 0$, the vanishing of D shows that the equation represents straight lines, which now are parallel. To prove this, write the equation so that a is positive. Write for a, λ^2 and for h, $\lambda\mu$; then since $ab = h^2$, for b must be written μ^2. The condition $D = 0$, that is—

$$c(ab - h^2) + 2fgh - af^2 - bg^2 = 0,$$

becomes
$$2fgh - af^2 - bg^2 = 0,$$
that is,
$$-\lambda^2 f^2 + 2\lambda\mu fg - \mu^2 g^2 = 0,$$
or
$$(\lambda f - \mu g)^2 = 0.$$

Hence $\dfrac{g}{\lambda} = \dfrac{f}{\mu}$; write k for the value of each of these fractions, then $g = k\lambda$, $f = k\mu$. The given equation is now of the form—

$$\lambda^2 x^2 + 2\lambda\mu xy + \mu^2 y^2 + 2k\lambda x + 2k\mu y + c = 0,$$

that is,
$$(\lambda x + \mu y)^2 + 2k(\lambda x + \mu y) + c = 0.$$

The left-hand side is a quadratic expression in $(\lambda x + \mu y)$, hence it is the product of two factors—

$$\lambda x + \mu y + \nu_1, \ \lambda x + \mu y + \nu_2;$$

the equation is therefore—

$$(\lambda x + \mu y + \nu_1)(\lambda x + \mu y + \nu_2) = 0,$$

which represents the two parallel lines—

$$\lambda x + \mu y + \nu_1 = 0,$$
$$\lambda x + \mu y + \nu_2 = 0.$$

EXAMPLES.

1. Find for each of the following equations the value of λ which will make the equation represent straight lines—

$$\text{(i) } \lambda x^2 + 4xy - y^2 + 6x + 10y + 1 = 0.$$
$$\text{(ii) } 3x^2 + 5xy + y^2 + 7x + 2y + \lambda = 0.$$
$$\text{(iii) } 9x^2 + 6xy + y^2 + \lambda x + 2y - 2 = 0.$$
$$\text{(iv) } 4x^2 - 3xy - y^2 + \lambda x + y + 2 = 0.$$

2. Prove that if $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents two parallel lines, these lines are parallel to each of

$$ax + hy + g = 0,\ hx + by + f = 0.$$

3. Prove that if the origin is half-way between a pair of parallel lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, then $g = 0$ and $f = 0$.

155. (ii) If the axes are turned through an angle θ, the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

becomes—

$$a(x\cos\theta - y\sin\theta)^2 + 2h(x\cos\theta - y\sin\theta)(x\sin\theta + y\cos\theta)$$
$$+ b(x\sin\theta + y\cos\theta)^2$$
$$+ 2g(x\cos\theta - y\sin\theta) + 2f(x\sin\theta + y\cos\theta) + c = 0,$$

that is, $$a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c' = 0,$$

where—

$$a' = a\cos^2\theta + b\sin^2\theta + 2h\sin\theta\cos\theta,$$
$$b' = a\sin^2\theta + b\cos^2\theta - 2h\sin\theta\cos\theta,$$
$$2h' = -2a\sin\theta\cos\theta + 2b\sin\theta\cos\theta + 2h(\cos^2\theta - \sin^2\theta);$$
$$2g' = 2g\cos\theta + 2f\sin\theta,\quad 2f' = -2g\sin\theta + 2f\cos\theta,\quad c' = c.$$

Notice that a', b', h' are expressed in terms of a, b, h, and θ; f', g' in terms of f, g, and θ. Thus the coefficients of terms of different degrees are kept separate.

From the values found for a', b', h' we obtain, in virtue of the trigonometrical identities,

$$\cos^2\theta+\sin^2\theta=1,\ \cos^2\theta-\sin^2\theta=\cos 2\theta,\ 2\sin\theta\cos\theta=\sin 2\theta,$$

the more convenient relations

$$\begin{aligned} a' + b' &= a + b, \\ a' - b' &= (a - b)\cos 2\theta + 2h\sin 2\theta, \\ 2h' &= -(a - b)\sin 2\theta + 2h\cos 2\theta. \end{aligned}$$

From the last two, by squaring and adding, we find—

$$(a' - b')^2 + 4h'^2 = (a - b)^2 + 4h^2.$$

But $$(a' + b')^2 = (a + b)^2,$$

hence by subtraction,

$$4a'b' - 4h'^2 = 4ab - 4h^2,$$

that is, $$a'b' - h'^2 = ab - h^2.$$

Thus when an equation of the second degree is transformed by a change of rectangular axes, the values of $a + b$ and $ab - h^2$ are unaffected. These two expressions, now shown to be invariable, are called *invariants* of the expression $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ for these transformations.

If the new axes are at our disposal, we may conveniently choose θ so as to simplify the equation by making one coefficient zero. Since θ enters into the coefficients a', b', h' of the terms of highest degree, it is natural to choose one of these to be made zero, and symmetry suggests that we get rid of the term xy rather than of x^2

or y^2. We therefore give to θ a value that will make $h' = 0$; that is, we choose θ to satisfy the equation—

$$-(a-b)\sin 2\theta + 2h\cos 2\theta = 0,$$

that is,

$$\tan 2\theta = \frac{2h}{a-b}.$$

The tangent of an angle assumes all positive and all negative values, once, as the angle ranges from 0 to π; hence there certainly is one positive value for 2θ, $< \pi$, that satisfies this equation. The half of this is $< \frac{\pi}{2}$; hence there is *one acute angle* that satisfies the equation.

Note.—The equation has of course an indefinite number of solutions, $2\theta = n\pi + \alpha$, where α is any one solution, and n any

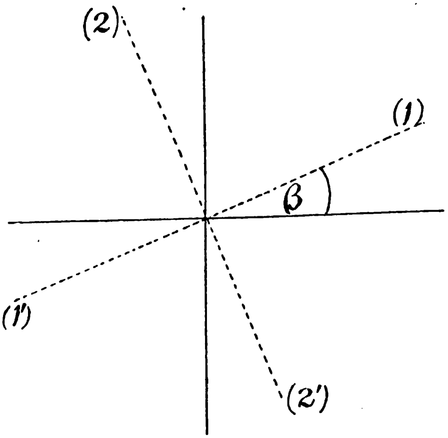

Fig. 87.

integer, positive or negative. For definiteness, take for α the value $< \pi$. Write $\alpha = 2\beta$; then $\theta = n\frac{\pi}{2} + \beta$.

If $n = 4k$, the line through O determined by θ coincides with (1) (Fig. 87); if $n = 4k + 1$, the line coincides with (2); if $n = 4k + 2$, the line coincides with (1′), the continuation of (1); and if $n = 4k + 3$, with (2′), the continuation of (2). Hence all the possible solutions indicate only two lines, at right angles; one of these we take as axis of x, the other as axis of y. There is nothing to be gained by attending to the general solution of the equation $\tan 2\theta = \frac{2h}{a - b}$, for there is no loss of generality in using only one particular solution. We take therefore the simplest solution available, and choose for θ the acute angle.

Thus if the axes are turned through a certain acute angle, determined by the equation $\tan 2\theta = \frac{2h}{a - b}$, the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

is reduced to the simpler form—

$$a'x^2 + b'y^2 + 2g'x + 2f'y + c' = 0,$$

where the values of a', b', f', g' are found by calculation from the formulæ—

$$\begin{aligned} a' + b' &= a + b, \\ a' - b' &= (a - b)\cos 2\theta + 2h \sin 2\theta, \\ g' &= g\cos\theta + f\sin\theta, f' = -g\sin\theta + f\cos\theta. \end{aligned}$$

For the calculation of a' and b', only $\sin 2\theta$ and $\cos 2\theta$ are needed. Since 2θ lies between 0 and π, $\sin 2\theta$ is positive, and $\cos 2\theta$ has the same sign as $\tan 2\theta$. Hence a' and b' are determined without ambiguity. Their values are most conveniently found by the following formulæ.

Since $\tan 2\theta = \frac{2h}{a - b}$, $\quad \sin 2\theta = \frac{2h}{\sqrt{(a - b)^2 + 4h^2}}$,

$\cos 2\theta = \dfrac{a-b}{\sqrt{(a-b)^2+4h^2}}$, where the sign of the radical is to be chosen so as to make $\sin 2\theta$ positive. Write R for this radical taken with sign the same as that of $2h$, then—

$$\sin 2\theta = \frac{2h}{\mathrm{R}}, \quad \cos 2\theta = \frac{a-b}{\mathrm{R}}.$$

Hence
$$\begin{aligned} a' - b' &= (a-b)\cos 2\theta + 2h \sin 2\theta \\ &= \frac{(a-b)^2+4h^2}{\mathrm{R}} = \frac{\mathrm{R}^2}{\mathrm{R}} = \mathrm{R}. \end{aligned}$$

The coefficients a', b' are therefore determined by—

$$\begin{aligned} a' + b' &= a + b, \\ a' - b' &= \sqrt{(a-b)^2+4h^2}, \end{aligned}$$

taken to be of the same sign as $2h$.

For the calculation of f' and g', $\sin\theta$ and $\cos\theta$ are needed; $\tan\theta$ is known from the equation—

$$\tan 2\theta = \frac{2h}{a-b}, \text{ that is, } \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2h}{a-b},$$

or
$$\tan^2\theta + \frac{a-b}{h}\tan\theta - 1 = 0.$$

Since θ is an acute angle, the value of $\tan\theta$ is given by the positive root of this equation, and from this $\sin\theta$ and $\cos\theta$ are known. Inasmuch as these are both positive (since θ is acute), there is no ambiguity in the determination of f' and g'. If in the first instance the origin has been changed so as to deprive the equation of the terms of the first degree, that is, if $f = 0$ and $g = 0$, then also $f' = 0$ and $g' = 0$, hence there is no calculation of f', g' to be performed.

The possible vanishing of $ab - h^2$ does not interfere with the determination of θ; but inasmuch as $a'b' - h'^2 = ab - h^2$, if $ab - h^2 = 0$ and $h' = 0$, it follows that $a'b' = 0$, from which either $a' = 0$ or $b' = 0$; hence in depriving the equation of the term xy we deprive it also of one of the two terms x^2, y^2.

Thus by turning the axes through a certain angle, we can reduce the equation of the second degree to the form—

$$ax^2 + by^2 + 2gx + 2fy + c = 0, \text{ if } ab - h^2 \neq 0,$$

and to the form $by^2 + 2gx + 2fy + c = 0$,

or $$ax^2 + 2gx + 2fy + c = 0, \text{ if } ab - h^2 = 0.$$

156. By the processes of §§ 153 and 155 the equation of the second degree is reduced to its simplest form.

I. If $ab - h^2 \neq 0$, first change the origin to the point given by $ax + hy + g = 0$, $hx + by + f = 0$, thus reducing the equation to $ax^2 + 2hxy + by^2 + c' = 0$.

If $c' = 0$, the equation represents straight lines, which may be found separately by factoring $ax^2 + 2hxy + by^2$.

If $c' \neq 0$, turn the axes through the acute angle θ that is determined by $\tan 2\theta = \dfrac{2h}{a - b}$; this reduces the equation to $a'x^2 + b'y^2 + c' = 0$.

(1) If a' and b' have the same sign, unlike that of c', this equation is $Ax^2 + By^2 = 1$, where $A\left(= -\dfrac{a'}{c'}\right)$ and $B\left(= -\dfrac{b'}{c'}\right)$ are positive. This represents an ellipse.

If a' and b' have the same sign, like that of c', the locus of the equation is entirely imaginary; it is called an imaginary ellipse.

(2) If a' and b' have opposite signs, the equation takes one of the forms—

$$Ax^2 - By^2 = 1, \quad -Ax^2 + By^2 = 1,$$

where both A and B are positive. It represents a hyperbola, whose major axis lies along the axis of x in the first case, along the axis of y in the second case.

Since $a'b' = ab - h^2$, if $ab - h^2$ is positive $a'b'$ is positive also, hence a' and b' have the same sign; if $ab - h^2$ is negative, a' and b' have opposite signs. Hence the general equation of the second degree represents an ellipse (or else a pair of imaginary straight lines) if $ab - h^2$ is positive; it represents a hyperbola (or else a pair of real straight lines) if $ab - h^2$ is negative.

II. If $ab - h^2 = 0$, change the direction of the axes in the first place; this reduces the equation to—

$$b'y^2 + 2g'x + 2f'y + c' = 0$$
$$(\text{or else } a'x^2 + 2g'x + 2f'y + c' = 0).$$

Then change the origin to (p, q). The equation becomes—

$$b'(y^2 + 2qy + q^2) + 2g'(x + p) + 2f'(y + q) + c' = 0,$$

that is,

$$b'y^2 + 2g'x + 2(b'q + f')y + b'q^2 + 2g'p + 2f'q + c' = 0.$$

To simplify this, choose p, q to satisfy—

$$b'q + f' = 0,$$
$$b'q^2 + 2g'p + 2f'q + c' = 0;$$

that is, take $q = -\frac{f'}{b'}$, $p = \frac{f'^2 - b'c'}{2b'g'}$; by this (unless $g' = 0$, when the equation is $b'y^2 + 2f'y + c' = 0$, which

represents parallel straight lines, real or imaginary), the equation is reduced to $b'y^2 + 2g'x = 0$, which represents a parabola. Hence if $ab - h^2 = 0$, that is, if the terms of the second degree are a perfect square, the equation of the second degree represents a parabola, or else a pair of parallel straight lines.

This completes the proof that an equation of the second degree represents either a conic or (as a particular case) a pair of straight lines.

EXAMPLES.

157. *Example* i.—Find and trace the curve represented by the equation—

$$7x^2 - 8xy + y^2 + 10x + 2y - 44 = 0.$$

Since $ab - h^2 = 7 - 16 = -9$, the curve is a hyperbola, or else a pair of real straight lines.

Take as origin the point given by—

$$7x - 4y + 5 = 0,$$
$$-4x + y + 1 = 0,$$

that is, the point (1, 3). The new absolute term is—

$$c' = 5 \times 1 + 1 \times 3 - 44 = -36.$$

The equation, transformed by this change of origin, is—

$$7x^2 - 8xy + y^2 - 36 = 0.$$

Turn the axes through the acute angle θ, determined by—

$$\tan 2\theta = \frac{-8}{6} = -\frac{4}{3};$$

the values of a', b' are given by the equations—

$$a' + b'(= a + b) = 8$$
$$a' - b' = -\sqrt{(a-b)^2 + 4h^2} = -10.$$

Hence $\qquad a' = -1, \ b' = 9.$

The transformed equation is therefore—

$$-x^2 + 9y^2 - 36 = 0,$$

that is, $\qquad -x^2 + 9y^2 = 36.$

This shows that the curve represented is the hyperbola—

$$-\frac{x^2}{36}+\frac{y^2}{4}=1,$$

which has the axis of y as transverse axis.

Before drawing this, it is necessary to put in the final axes correctly with reference to the original axes. These final axes pass through the point (1, 3), and make with the original axes the acute angle θ determined by—

$$\tan 2\theta = -\frac{4}{3}.$$

Hence

$$\frac{2\tan\theta}{1-\tan^2\theta}=-\frac{4}{3},$$

$$2\tan^2\theta - 3\tan\theta - 2 = 0,$$

$$\tan\theta = 2 \text{ or } -\frac{1}{2}.$$

Since θ is an acute angle, the solution wanted is $\tan\theta = 2$. Hence

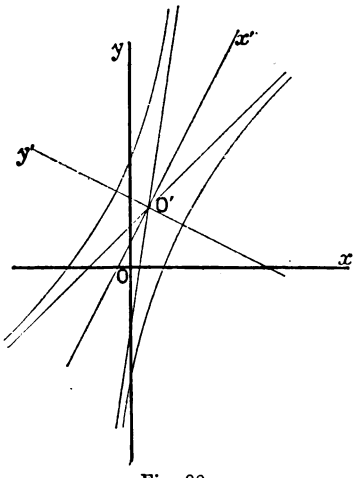

Fig. 88.

the final axes are as shown in Fig. 88, and the hyperbola is to be traced with reference to these. These axes can equally well be drawn by means of the value for $\tan 2\theta$.

It is well to test the result by finding, from the original equation, the points where the curve meets the original axes. The line $y = 0$ (the axis of x) meets $7x^2 - 8xy + y^2 + 10x + 2y - 44 = 0$ where

$$7x^2 + 10x - 44 = 0;$$

hence one value of x is positive, one negative; the curve meets the axis of x at two real points on opposite sides of the origin. The curve meets $x = 0$ where

$$y^2 + 2y - 44 = 0,$$

hence at two real points on opposite sides of the origin. The diagram agrees with these results.

Example ii.—Find and trace the curve represented by the equation—

$$5x^2 + 4xy + 2y^2 + 12y + 6 = 0.$$

Since $ab - h^2 = 6$, the curve is an ellipse, or else a pair of imaginary straight lines.

Take as origin the point given by—

$$5x + 2y = 0,$$
$$2x + 2y + 6 = 0,$$

that is, the point $(2, -5)$. The new absolute term is—

$$c' = 6 \times -5 + 6 = -24.$$

The equation, with this origin, is—

$$5x^2 + 4xy + 2y^2 - 24 = 0.$$

Turn the axes through the acute angle θ, determined by $\tan 2\theta = \dfrac{4}{3}$; the values of a', b' are given by the equations—

$$a' + b'(= a + b) = 7,$$
$$a' - b'(= + \sqrt{(a - b)^2 + 4h^2}) = 5.$$

Hence $$a' = 6,\ b' = 1.$$

The transformed equation is therefore—

$$6x^2 + y^2 - 24 = 0,$$

that is, $$\frac{x^2}{4} + \frac{y^2}{24} = 0,$$

which represents an ellipse, with its major axis along the axis of y.

Since $$\tan 2\theta = \frac{4}{3},$$

$$2\tan^2\theta + 3\tan\theta - 2 = 0,$$

hence $$\tan\theta = \frac{1}{2};$$

the final axes pass through the point (2, − 5), and make with the original axes the angle $\tan^{-1}\frac{1}{2}$. These axes and the ellipse lie as shown in Fig. 89. As a verification, the original axis of x meets the curve where $5x^2 + 6 = 0$, that is, at imaginary points; and the original axis of y meets it where $2y^2 + 12y + 6 = 0$, that is, where $y = -3 \pm \sqrt{6}$.

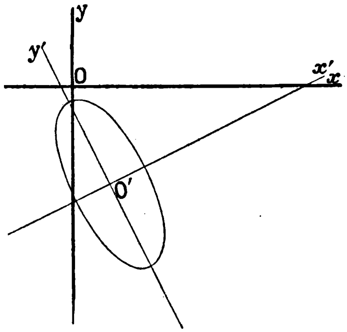

Fig. 89.

Example iii.—Find and trace the curve represented by the equation—

$$4x^2 + 4xy + y^2 + 6x - 2y - 1 = 0.$$

Since $ab - h^2 = 0$, begin by turning the axes through the acute angle θ determined by $\tan 2\theta = \frac{4}{3}$; this equation gives $\tan\theta = \frac{1}{2}$ hence $\sin\theta = \frac{1}{\sqrt{5}}$, $\cos\theta = \frac{2}{\sqrt{5}}$.

The values of a', b' are given by—

$$a' + b' = 5,$$

$$a' - b = +\sqrt{3^2 + 4^2} = +5,$$

hence $$a' = 5,\ b = 0.$$

The values of f, g' are given by—

$$g' = g\cos\theta + f\sin\theta = \frac{6}{\sqrt{5}} - \frac{1}{\sqrt{5}} = \sqrt{5},$$

$$f = -g\sin\theta + f\cos\theta = -\frac{3}{\sqrt{5}} - \frac{2}{\sqrt{5}} = -\sqrt{5}.$$

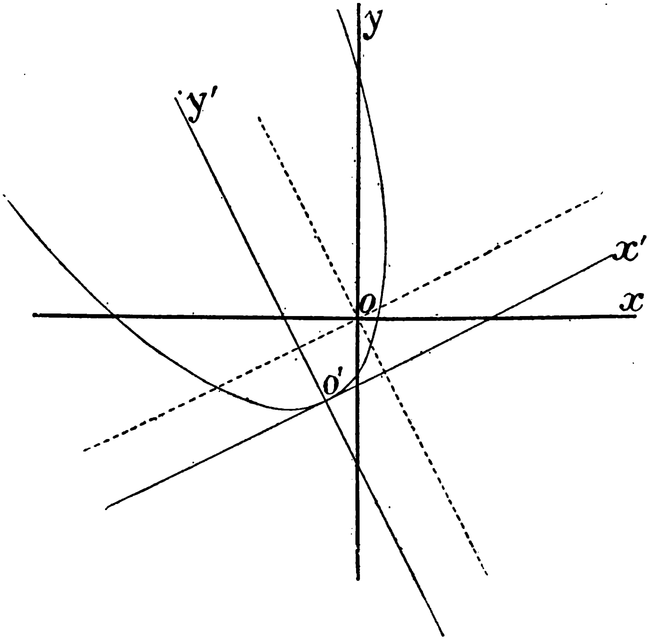

Fig. 90.

The equation is now –

$$5x^2 + 2\sqrt{5}\,x - 2\sqrt{5}\,y - 1 = 0.$$

Next change the origin to p, q. The equation becomes—

$$5(x^2 + 2px + p^2) + 2\sqrt{5}\,(x + p) - 2\sqrt{5}\,(y + q) - 1 = 0,$$

that is—

$$5x^2 + 2(5p + \sqrt{5})x - 2\sqrt{5}\,y + 5p^2 + 2\sqrt{5}\,p - 2\sqrt{5}\,q - 1 = 0.$$

Choose p, q to satisfy—

$$5p + \sqrt{5} = 0,$$
$$5p^2 + 2\sqrt{5}p - 2\sqrt{5}q - 1 = 0,$$

that is, take as origin the point $\left(-\frac{1}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right)$. The transformed equation is—

$$5x^2 - 2\sqrt{5}y = 0,$$

that is,
$$x^2 = \frac{2}{\sqrt{5}}y,$$

which represents a parabola, whose axis lies along the axis of y. In Fig. 90 the three sets of axes and the parabola are represented; notice that the coordinates of the final origin are found with reference to the intermediate axes.

The curve meets the original axis of x where $4x^2 + 6x - 1 = 0$, and the original axis of y where $y^2 - 2y - 1 = 0$; the diagram agrees with these results.

158. A different process of reduction is however better in the case of the parabola. It is known that the equation can be reduced to $y'^2 = 4px'$ where y' is the distance from the point whose (original) coordinates are x, y to a line whose (original) equation is $lx + my + k = 0$, and x' is the distance from the point to a perpendicular line, $l'x + m'y + k' = 0$, these two lines being the final axes. Hence the given equation must be of the form—

$$(lx + my + k)^2 = \mathrm{M}(l'x + m'y + k'),$$

with the condition which expresses that the two lines are perpendicular, $ll' + mm' = 0$.

Write the given equation—

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

so that the coefficient a is positive; take quantities l, m

determined by $a = l^2$, $h = lm$; since $ab = h^2$, these give $b = m^2$. The equation is therefore—

$$l^2x^2 + 2lmxy + m^2y^2 + 2gx + 2fy + c = 0,$$

that is,
$$(lx + my)^2 + 2gx + 2fy + c = 0.$$

Now introduce a quantity k (still to be determined); the equation can be written—

$$(lx + my + k)^2 = 2(kl - g)x + 2(km - f)y + k^2 - c.$$

This is of the proper form—

$$(lx + my + k)^2 = l'x + m'y + k',$$

and by means of the quantity k we can satisfy the condition $ll' + mm' = 0$. This requires—

$$l(kl - g) + m(km - f) = 0,$$

hence
$$k = \frac{lg + mf}{l^2 + m^2},$$

$$kl - g = \frac{lmf - m^2g}{l^2 + m^2} = \frac{lf - mg}{l^2 + m^2} \cdot m = \mathbf{M} \cdot m,$$

$$km - f = \frac{-l^2f + lmg}{l^2 + m^2} = \frac{lf - mg}{l^2 + m^2} \times -l = -\mathbf{M} \cdot l;$$

write the value of $k^2 - c$ as $\mathrm{M}k'$.

The equation is thus obtained in the form—

$$(lx + my + k)^2 = \mathbf{M}(mx - ly + k').$$

Now take as axes the perpendicular lines—

$$lx + my + k = 0,\ mx - ly + k' = 0.$$

If the first is the new axis of x, $\mathrm{O}x'$, then—

$$y' = \frac{lx + my + k}{\sqrt{l^2 + m^2}},\quad x' = \frac{mx - ly + k'}{\sqrt{l^2 + m^2}},$$

and the equation becomes—

$$(l^2 + m^2)y'^2 = M \cdot \sqrt{l^2 + m^2} \cdot x',$$

that is, $$y'^2 = 4px'.$$

Example.—The equation (§ 157, Ex. iii)—

$$4x^2 + 4xy + y^2 + 6x - 2y - 1 = 0,$$

is $$(2x + y)^2 + 6x - 2y - 1 = 0.$$

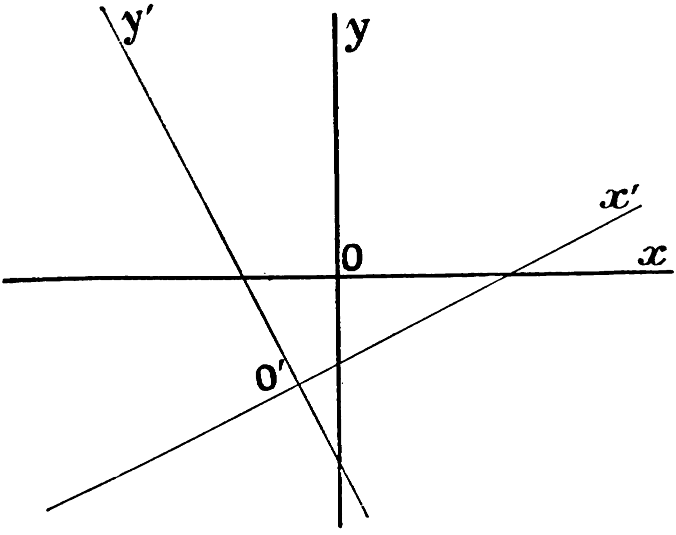

Fig. 91.

Write this $(2x + y + k)^2 = (4k - 6)x + (2k + 2)y + k^2 + 1$;

choose k to satisfy $2(4k - 6) + 2k + 2 = 0,$

that is, take for k the value 1. Then the equation is obtained in the form

$$\begin{aligned}(2x + y + 1)^2 &= -2x + 4y + 2 \\ &= -2(x - 2y - 1).\end{aligned}$$

Take as axes the two perpendicular lines—

$$2x + y + 1 = 0, \quad x - 2y - 1 = 0 \text{ (Fig. 91)}.$$

If $x - 2y - 1 = 0$ is taken as new axis of x, and $2x + y + 1 = 0$ as new axis of y, with the positive directions as shown in the figure, the formulæ of transformation are—

$$y' = -\frac{x - 2y - 1}{\sqrt{5}}, \quad x' = +\frac{2x + y + 1}{\sqrt{5}}.$$

Hence
$$x - 2y - 1 = -\sqrt{5}\cdot y',$$
$$2x + y + 1 = +\sqrt{5}\cdot x',$$

and the equation becomes
$$5x'^2 = 2\sqrt{5}\,y',$$

that is, as before,
$$x^2 = \frac{2}{\sqrt{5}}y.$$

159. *Centre of a conic.*—In § 156 it was proved that unless $ab - h^2 = 0$, the general equation of the second degree can be reduced to the form $a'x^2 + b'y^2 + c' = 0$, and therefore represents an ellipse or hyperbola, whose centre is the point there indicated as the suitable origin. Hence the centre of a conic—

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

is the point of intersection of the two lines—

$$ax + hy + g = 0, \quad hx + by + f = 0;$$

the coordinates of the centre are $\dfrac{hf - bg}{ab - h^2}, \dfrac{gh - af}{ab - h^2}$.

Asymptotes.—In the transformation of the equation, the expression $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ becomes first $ax^2 + 2hxy + by^2 + c'$, and then $a'x^2 + b'y^2 + c'$. Now the asymptotes of the conic $\mathrm{A}x^2 + \mathrm{B}y^2 - 1 = 0$ are $\mathrm{A}x^2 + \mathrm{B}y^2 = 0$; hence the asymptotes of $a'x^2 + b'y^2 + c' = 0$ are $a'x^2 + b'y^2 = 0$; that is, if $v = 0$ is the equation of the curve in its final form, the asymptotes are $v - c' = 0$,

where $c' = \dfrac{D}{ab - h^2}$. Now retrace the process of transformation: v, that is, $a'x^2 + b'y^2 + c'$, becomes—

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c,$$

hence $v - c'$ becomes $ax^2 + 2hxy + by^2 + 2gx + 2fy + c - c'$. That is, the asymptotes of the conic—

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

are $ax^2 + 2hxy + by^2 + 2gx + 2fy + c - \dfrac{D}{ab - h^2} = 0.$

This result is most easily remembered in the form—The equation of the asymptotes of a conic differs from the equation of the conic itself only in the absolute term. Hence to find the asymptotes, write λ instead of the given absolute term, and determine λ so that the equation thus modified shall represent straight lines.

If $a + b = 0$, the asymptotes are at right angles, hence the conic is a rectangular hyperbola.

Example.—Find the asymptotes of—

$$7x^2 - 8xy + y^2 + 10x + 2y - 44 = 0.$$

The equation $7x^2 - 8xy + y^2 + 10x + 2y + \lambda = 0$ represents straight lines if (§ 154)—

$$7\lambda - 40 - 7 - 25 - 16\lambda = 0,$$

that is, if $\lambda = -8.$

Hence the asymptotes are—

$$7x^2 - 8xy + y^2 + 10x + 2y - 8 = 0,$$

that is, $(7x - y - 4)(x - y + 2) = 0.$

Note.—The equation of the curve can be written—

$$(7x - y - 4)(x - y + 2) = 36. \quad \text{(Compare § 138.)}$$

160. *Foci and directrices of a central conic.*—After the equation has been reduced, the coordinates of the foci

with reference to the final axes are known. Also, by means of the equations of transformation, the original coordinates are expressed in terms of the final coordinates. Hence the original coordinates of the foci can be found. Similarly from the equations of the directrices and the axes of the curve, with respect to the final axes, there can be found the equations with respect to the original axes.

Example i.—In § 157 it was shown that the equation—

$$7x^2 - 8xy + y^2 + 10x + 2y - 44 = 0,$$

can be reduced to
$$-\frac{x^2}{36} + \frac{y^2}{4} = 1,$$

by first changing the origin to (1, 3), and then turning the axes through the angle $\tan^{-1} 2$. If the original, the intermediate, and the final coordinates of a point are represented by (x, y), (x_1, y_1), (x', y'), the formulæ of transformation are—

$$x = x_1 + 1 = \frac{1}{\sqrt{5}}x' - \frac{2}{\sqrt{5}}y' + 1,$$

$$y = y_1 + 3 = \frac{2}{\sqrt{5}}x' + \frac{1}{\sqrt{5}}y' + 3,$$

from which
$$\sqrt{5}\,x' = x + 2y - 7,$$
$$\sqrt{5}\,y' = -2x + y - 1.$$

The foci of the hyperbola $-\frac{x^2}{36} + \frac{y^2}{4} = 1$ are on the axis of y', at a distance $\sqrt{4 + 36}$ from the centre; their final coordinates are therefore $(0, \pm 2\sqrt{10})$. Hence the original coordinates of one focus are—

$$x = -\frac{2}{\sqrt{5}} . 2\sqrt{10} + 1 = -4\sqrt{2} + 1,$$

$$y = \frac{1}{\sqrt{5}} . 2\sqrt{10} + 3 = 2\sqrt{2} + 3;$$

the two foci are $(-4\sqrt{2} + 1, 2\sqrt{2} + 3)$ and $(4\sqrt{2} + 1, -2\sqrt{2} + 3)$.

The major axis is 4; hence $e = \frac{4\sqrt{10}}{4} = \sqrt{10}$. The final equa-

tions of the directrices are $y' = \pm \dfrac{2}{\sqrt{10}}$, hence their original equations are—

$$-2x + y - 1 = \pm\sqrt{5} \times \frac{2}{\sqrt{10}},$$

or $$2x - y + 1 \pm \sqrt{2} = 0.$$

The equation of the transverse axis is $x' = 0$; the equation referred to the original axes is therefore $x + 2y - 7 = 0$, and the equation of the conjugate axis is $2x - y + 1 = 0$.

Example ii.—The equation $5x^2 + 4xy + 2y^2 + 12y + 6 = 0$ is reduced to $$\frac{x^2}{4} + \frac{y^2}{24} = 1 \text{ (Ex. ii, § 157).}$$

Exactly as above, it is found that the foci of this ellipse are $(0, -1)$ and $(4, -9)$; the eccentricity is $\sqrt{\dfrac{5}{6}}$; the directrices are—

$$x - 2y = 0, \; x - 2y - 24 = 0;$$

the transverse axis is $2x + y + 1 = 0$, and the conjugate axis is $x - 2y - 12 = 0$.

161. *Focus and directrix of the parabola.*—The equation of the parabola can be written in the form—

$$(lx + my + k)^2 = \text{M}(mx - ly + k')^2;$$

the axis of the curve is the line $lx + my + k = 0$, the vertex is the intersection of this with $mx - ly + k' = 0$, which is the tangent at the vertex. The focus and directrix are most easily found by the process of the following example.

Example.—Find the focus, directrix, and latus rectum of the parabola—

$$4x^2 + 4xy + y^2 + 6x - 2y - 1 = 0.$$

The equation is $$(2x + y)^2 + 6x - 2y - 1 = 0.$$

Hence the axis is parallel to $2x + y = 0$; and the directrix is therefore perpendicular to $2x + y = 0$. The equation of the directrix is

consequently $x - 2y + \lambda = 0$, where λ is still to be determined. If the focus is (x', y'), the equation of the curve is—

$$[(x - x')^2 + (y - y')^2]^{\frac{1}{2}} = \frac{x - 2y + \lambda}{\sqrt{5}},$$

that is, $$5[(x - x')^2 + (y - y')^2] = (x - 2y + \lambda)^2,$$

or $$4x^2 + 4xy + y^2 - (10x' + 2\lambda)x - (10y' - 4\lambda)y + 5x'^2 + 5y'^2 - \lambda^2 = 0.$$

Comparison of this form with the given equation shows that—

$$10x' + 2\lambda = -6,$$
$$10y' - 4\lambda = 2,$$
$$5x'^2 + 5y'^2 - \lambda^2 = -1.$$

The first two of these equations give—

$$x' = -\frac{\lambda + 3}{5}, \; y' = \frac{2\lambda + 1}{5},$$

hence, from the third,

$$(\lambda + 3)^2 + (2\lambda + 1)^2 - 5\lambda^2 = -5,$$
$$10\lambda = -15, \; \lambda = -\frac{3}{2},$$

and therefore $$x = -\frac{3}{10}, \; y' = -\frac{2}{5}.$$

The focus of the parabola is $\left(-\frac{3}{10}, -\frac{2}{5}\right)$, and the directrix is $x - 2y - \frac{3}{2} = 0$. The latus rectum is equal to twice the distance from the focus to the directrix, hence—

$$l.r. = 2 \times -\frac{-\frac{3}{10} + \frac{4}{5} - \frac{3}{2}}{\sqrt{5}} = \frac{2}{\sqrt{5}}.$$

Note.—Since the equation can be reduced to $x^2 = \frac{2}{\sqrt{5}}y$, we know in another manner that the latus rectum is $\frac{2}{\sqrt{5}}$.

EXAMPLES.

Reduce each of the following equations to its simplest form; draw the curve that it represents.

Give the coordinates of the centre and the foci referred to the

original axes, also the equations of the directrices and the axes of each curve, stating which is the transverse axis.

If the curve is a hyperbola, give the equation of the asymptotes referred to the original axes.

1. $7x^2 - 8xy + y^2 + 10x + 2y - 44 = 0$.
2. $5x^2 + 4xy + 2y^2 + 12y + 6 = 0$.
3. $4x^2 + 4xy + y^2 - 6x + 10y - 1 = 0$.
4. $61x^2 - 64xy + 109y^2 + 64x - 218y - 1016 = 0$.
5. $28x^2 - 24xy + 21y^2 + 112x - 48y - 36 = 0$.
6. $16x^2 + 24xy + 9y^2 - 156x + 8y - 96 = 0$.
7. $7x^2 + 24xy - 148x - 48y + 124 = 0$.
8. $5x^2 - 24xy - 5y^2 - 280x - 4y + 62 = 0$.
9. $79x^2 + 60xy + 254y^2 - 80x - 332y + 77 = 0$.
10. $65x^2 - 108xy - 250y^2 + 32x + 340y + 1 = 0$.
11. $9x^2 + 12xy + 4y^2 + 13y - 23 = 0$.
12. $144x^2 + 120xy + 25y^2 - 5x + 12y + 169 = 0$.

162. It has been shown (§ 154) that the general equation of the second degree represents straight lines if

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$$

This condition can be found in another manner. If the equation represents straight lines, it can be made to yield relations of the form $y = mx + n$. If then we treat it as a quadratic in y, which we can do unless $b = 0$, it must be possible to carry on the work to the end, obtaining a result linear in x, y.

The equation, written as a quadratic in y, is—

$$by^2 + 2(hx + f)y = -(ax^2 + 2gx + c),$$

that is—

$$\begin{aligned}(by + hx + f)^2 &= (hx + f)^2 - b(ax^2 + 2gx + c)\\ &= (h^2 - ab)x^2 + 2(hf - bg)x + f^2 - bc.\end{aligned}$$

Hence—

$$by = -(hx + f) \pm [(h^2 - ab)x^2 + 2(hf - bg)x + f^2 - bc]^{\frac{1}{2}}.$$

This shows that y is a linear function of x, only if the expression under the radical sign is a perfect square. The condition is therefore—

$$(hf - bg)^2 = (h^2 - ab)(f^2 - bc),$$
$$h^2f^2 - 2bfgh + b^2g^2 = h^2f^2 - abf^2 - bch^2 + ab^2c,$$

that is, after division by b,

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$$

If this condition is satisfied, the expression under the radical sign is a perfect square, $(px + q)^2$; hence—

$$by = -(hx + f) \pm (px + q),$$

and the equation represents these two straight lines.

Note.—It was proved in § 37 that the angle between the lines is $\tan^{-1} \dfrac{2\sqrt{h^2 - ab}}{a + b}$.

Example.—Find the lines represented by—

$$2x^2 - 5xy - 3y^2 + 8x + 11y - 10 = 0.$$

First solution.—

$$3y^2 + (5x - 11)y = 2x^2 + 8x - 10$$
$$9y^2 + 3(5x - 11)y = 6x^2 + 24x - 30$$
$$\left(3y + \frac{5x - 11}{2}\right)^2 = \left(\frac{5x - 11}{2}\right)^2 + 6x^2 + 24x - 30$$
$$= \frac{1}{4}(49x^2 - 14x + 1)$$
$$= \frac{1}{4}(7x - 1)^2.$$
$$3y = \frac{-(5x - 11) \pm (7x - 1)}{2}.$$
$$= x + 5 \text{ or } -6x + 6.$$

The lines represented by the equation are $x - 3y + 5 = 0$, $2x + y - 2 = 0$.

Second solution.—The factors can be found as conveniently by a different method. The terms of the second degree can be factored by inspection; they are $(x - 3y)(2x + y)$. If the whole expression has linear factors, they must be $x - 3y + p$, $2x + y + q$. Hence $(x - 3y + p)(2x + y + q)$ must be the same as the given expression. This product is—

$$2x^2 - 5xy - 3y^2 + (2p + q)x + (p - 3q)y + pq;$$

hence
$$2p + q = 8,$$
$$p - 3q = 11,$$

from which $p = 5$, $q = -2$; and inasmuch as these values give to pq the necessary value -10 (the absolute term in the given equation), the equation represents the two lines $x - 3y + 5 = 0$, $2x + y - 2 = 0$.

163. In certain cases the form of an equation of degree n indicates that it represents straight lines—

1. If the equation involves only one of the coordinates, it represents n straight lines parallel to an axis. For by the principles of the theory of equations,

$$ax^n + bx^{n-1} + cx^{n-2} + \dots = 0$$

is equivalent to

$$a(x - x_1)(x - x_2) \dots (x - x_n) = 0.$$

2. If the equation is homogeneous, it represents n straight lines through the origin (§ 64).

164. The lines that bisect the angles between

$$ax^2 + 2hxy + by^2 = 0$$

are given by the equation
$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}.$$

To prove this, denote the angles that the two given lines make with Ox by θ_1, θ_2, and the angle that a bisector makes with Ox by θ. Then $\theta = \frac{\theta_1 + \theta_2}{2}$, and for the other bisector $\theta = \frac{\pi}{2} + \frac{\theta_1 + \theta_2}{2}$, hence $2\theta = \theta_1 + \theta_2$ or $\pi + \theta_1 + \theta_2$; consequently for each bisector—

$$\begin{aligned}\tan 2\theta &= \tan(\theta_1 + \theta_2)\\ &= \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1\tan\theta_2}\\ &= \frac{m_1 + m_2}{1 - m_1 m_2},\end{aligned}$$

where m_1, m_2 are the slopes of the two given lines. Now the equation $ax^2 + 2hxy + by^2 = 0$ is equivalent to $(y - m_1x)(y - m_2x) = 0$, hence $m_1 + m_2 = -\frac{2h}{b}$, $m_1m_2 = \frac{a}{b}$; consequently the directions of the bisectors are determined by—

$$\tan 2\theta = \frac{-\frac{2h}{b}}{1 - \frac{a}{b}} = \frac{2h}{a - b},$$

that is, by
$$\frac{2\tan\theta}{1 - \tan^2\theta} = \frac{2h}{a - b}.$$

If (x, y) is a point on either bisector, $\frac{y}{x} = \tan\theta$. Hence (x, y) must satisfy the equation—

$$\frac{2\frac{y}{x}}{1 - \frac{y^2}{x^2}} = \frac{2h}{a - b},$$

that is,
$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}.$$

This gives the equation of lines through the origin parallel to the axes of the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$. For the axes bisect the angles between the asymptotes; and the asymptotes are $ax^2 + 2hxy + by^2 + 2gx + 2fy + k = 0$, and are therefore parallel to $ax^2 + 2hxy + by^2 = 0$.

165. In § 68 it was proved that the point-equation of a conic is of the second degree, and in § 156 that every equation in point-coordinates of the second degree (that is not reducible) represents a conic. In § 87 it was proved that the line-equation of a conic is of the second degree; it is now to be proved that every irreducible equation of the second degree in line-coordinates represents a conic (as an envelope).

The line-equation of the general conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

can readily be found. The line

$$\xi x + \eta y + 1 = 0$$

meets this conic in two points; these are to be made indistinguishable. The equation of the two lines that join the origin to these points is (§ 64)—

$$ax^2 + 2hxy + by^2 - 2(gx + fy)(\xi x + \eta y) + c(\xi x + \eta y)^2 = 0,$$

that is,

$$(a - 2g\xi + c\xi^2)x^2 + 2(h - f\xi - g\eta + c\xi\eta)xy + (b - 2f\eta + c\eta^2)y^2 = 0.$$

The two lines are indistinguishable if

$$(c\xi^2 - 2g\xi + a)(c\eta^2 - 2f\eta + b) = (c\xi\eta - f\xi - g\eta + h)^2,$$

or

$$\begin{aligned} c^2\xi^2\eta^2 - 2cf\xi^2\eta - 2cg\xi\eta^2 + bc\xi^2 + 4fg\xi\eta + ca\eta^2 - 2bg\xi \\ - 2af\eta + ab \\ = c^2\xi^2\eta^2 - 2cf\xi^2\eta - 2cg\xi\eta^2 + f^2\xi^2 + 2fg\xi\eta \\ + 2ch\xi\eta + g^2\eta^2 - 2hf\xi - 2gh\eta + h^2, \end{aligned}$$

that is, if

$$(bc - f^2)\xi^2 + 2(fg - ch)\xi\eta + (ca - g^2)\eta^2 + 2(hf - bg)\xi + 2(gh - af)\eta + ab - h^2 = 0.$$

Hence the line-equation of the general conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

is $\quad \mathrm{A}\xi^2 + 2\mathrm{H}\xi\eta + \mathrm{B}\eta^2 + 2\mathrm{G}\xi + 2\mathrm{F}\eta + \mathrm{C} = 0,$

where

$$\begin{aligned} \mathrm{A} &= bc - f^2, & \mathrm{F} &= gh - af, \\ \mathrm{B} &= ca - g^2, & \mathrm{G} &= hf - bg, \\ \mathrm{C} &= ab - h^2; & \mathrm{H} &= fg - ch. \end{aligned}$$

Note.—The coefficients A, B, C, F, G, H are the co-factors of a, b, c, f, g, h in the determinant—

$$\mathrm{D} = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

By direct calculation, or by the properties of determinants, it can be shown that—

$$\begin{aligned} \mathrm{BC} - \mathrm{F}^2 &= a\mathrm{D}, & \mathrm{GH} - \mathrm{AF} &= f\mathrm{D}, \\ \mathrm{CA} - \mathrm{G}^2 &= b\mathrm{D}, & \mathrm{HF} - \mathrm{BG} &= g\mathrm{D}, \\ \mathrm{AB} - \mathrm{H}^2 &= c\mathrm{D}; & \mathrm{FG} - \mathrm{CH} &= h\mathrm{D}. \end{aligned}$$

These relations enable us to write down a point-equation of the second degree from which any given line-equation of the second degree is derived; they show therefore that a line-equation of the second degree represents a conic. (See also § 166.)

Example.—The line-equation

$$-18\xi^2 + 12\xi\eta + 5\eta^2 + 12\xi - 14\eta - 1 = 0$$

is derived from $\quad 3x^2 + 4xy + y^2 + 8x + 10y + 7 = 0.$

For

$$\begin{aligned} \mathrm{BC} - \mathrm{F}^2 &= -5 - 49 = -54, \\ \mathrm{CA} - \mathrm{G}^2 &= 18 - 36 = -18, \\ \mathrm{AB} - \mathrm{H}^2 &= -90 - 36 = -126; \\ \mathrm{GH} - \mathrm{AF} &= +36 - 126 = -90, \\ \mathrm{HF} - \mathrm{BG} &= -42 - 30 = -72, \\ \mathrm{FG} - \mathrm{CH} &= -42 + 6 = -36. \end{aligned}$$

Hence the point-equation is, save as to a numerical factor,

$$-54x^2 - 72xy - 18y^2 - 144x - 180y - 126 = 0,$$

that is, after rejection of the factor -18,

$$3x^2 + 4xy + y^2 + 8x + 10y + 7 = 0.$$

Note.—Since the condition for a parabola is $ab - h^2 = 0$, that is, $\mathrm{C} = 0$, the line-equation of a parabola is—

$$\mathrm{A}\xi^2 + 2\mathrm{H}\xi\eta + \mathrm{B}\eta^2 + 2\mathrm{G}\xi + 2\mathrm{F}\eta = 0.$$

This is satisfied by $\xi = 0$, $\eta = 0$, that is, by the coordinates of a straight line which lies entirely at infinity. This agrees with the conclusion arrived at in another manner in § 125.

The coefficients A, B, C, F, G, H occur constantly. For instance, the coordinates of the centre (§ 159) are $\frac{G}{C}$, $\frac{F}{C}$. Hence the equation of the centre is
$$\frac{G}{C}\xi + \frac{F}{C}\eta + 1 = 0,$$
that is,
$$G\xi + F\eta + C = 0.$$

If however the condition $D = 0$ is satisfied, so that the equation
$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$
represents a line pair, the three equations
$$\begin{aligned} ax + hy + g &= 0, \\ hx + by + f &= 0, \\ gx + fy + c &= 0, \end{aligned}$$
are satisfied by the intersection of the lines. Hence the coordinates of this point can be written in various forms; its equation can be given in any one of the forms—
$$\begin{aligned} A\xi + H\eta + G &= 0, \\ H\xi + B\eta + F &= 0, \\ G\xi + F\eta + C &= 0. \end{aligned}$$

166. The proof that a line-equation of the second degree represents a conic may be given as follows.

The line-equation of the conic
$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$
is found by expressing that the points common to this locus and the line (ξ, η), that is, the line $\xi x + \eta y + 1 = 0$, are indistinguishable. Similarly the point-equation of the envelope $A\xi^2 + 2H\xi\eta + B\eta^2 + 2G\xi + 2F\eta + C = 0$, may be found by expressing that the lines common to this and the point (x, y) are indistinguishable; the point is then the intersection of two consecutive lines of the system.

The equation of the point (x, y), that is, the condition to which the coordinates of a line are subject if the line passes through the point, is $\xi x + \eta y + 1 = 0$. The lines whose coordinates satisfy this and the given equation of the second degree satisfy also—

$$A\xi^2 + 2H\xi\eta + B\eta^2 - 2(G\xi + F\eta)(\xi x + \eta y) + C(\xi x + \eta y)^2 = 0,$$

that is,

$$(A - 2Gx + Cx^2)\xi^2 + 2(H - Fx - Gy + Cxy)\xi\eta + (B - 2Fy + Cy^2)\eta^2 = 0;$$

hence they are indistinguishable if

$$(Cx^2 - 2Gx + A)(Cy^2 - 2Fy + B) = (Cxy - Fx - Gy + H)^2$$

or

$$\begin{aligned} C^2x^2y^2 - 2CFx^2y - 2CGxy^2 + BCx^2 + 4FGxy + CAy^2 \\ - 2BGx - 2AFy + AB \\ = C^2x^2y^2 - 2CFx^2y - 2CGxy^2 + F^2x^2 + 2FGxy \\ + G^2y^2 + 2CHxy - 2HFx - 2GHy + H^2, \end{aligned}$$

that is, if

$$(BC - F^2)x^2 + 2(FG - CH)xy + (CA - G^2)y^2 + 2(HF - BG)x + 2(GH - AF)y + AB - H^2 = 0.$$

Hence from the line-equation of the second degree we obtain a point-equation, also of the second degree.

EXAMPLES.

1. Show that by a proper choice of axes, the equations of two parabolas whose axes are parallel can be reduced to the form—

$$y^2 + 2gx + 2fy + c = 0,$$
$$y^2 + 2kgx + 2kfy + k^2c = 0.$$

2. Hence show that the lines that join the points of contact of parallel tangents to two parabolas whose axes are parallel all pass through one point.

3. A parabola passes through two given points, and its axis is parallel to a given line. Prove that the locus of the focus is a hyperbola, whose foci are the two given points.

4. Show that the equation $\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} = 1$ represents a parabola, which touches the axes of coordinates. Find the focus and directrix.

5. Show that $(\lambda x + \mu y + \nu)^{\frac{1}{2}} + (\lambda' x + \mu' y + \nu')^{\frac{1}{2}} = k$ represents a parabola.

6. Show that

$$(\lambda x + \mu y + \nu)^{\frac{1}{2}} + (\lambda' x + \mu' y + \nu')^{\frac{1}{2}} + (\lambda'' x + \mu'' y + \nu'')^{\frac{1}{2}} = 0$$

represents a conic, which touches the three lines $\lambda x + \mu y + \nu = 0$, $\lambda' x + \mu' y + \nu' = 0$, $\lambda'' x + \mu'' y + \nu'' = 0$.

7. Find lines through the origin parallel to the axes of the conics

$$\text{(i)}\ \ 13x^2 + 5xy + y^2 + 7x - 2y - 1 = 0,$$

$$\text{(ii)}\ \ 9x^2 - 12xy + 4y^2 + 4x + 5y + 2 = 0.$$

8. Find the line-equation of $4x^2 - 5xy + y^2 - 7x + 9y + 1 = 0$, and the point-equation of $11\xi^2 + 18\xi\eta - 17\eta^2 - 32\xi - 14\eta + 5 = 0$.

9. Write down the equation and the coordinates of the centre of each of the conics in Ex. 8.

10. Prove that a line-equation of the second degree in which the absolute term is zero represents a parabola.

11. Find the condition to be satisfied by the coefficients of the equation $ax^3 + bx^2y + cxy^2 + dy^3 = 0$, in order that two of the lines may be at right angles.

12. Show that three lines through the origin, equally inclined, are given by $x^3 - 3xy^2 = \lambda(3x^2y - y^3)$.

13. Prove that any tangent to an ellipse meets the orthocycle (director circle, § 83) on conjugate diameters of the ellipse.

14. Show that for all values of ϕ, the line

$$x(e + \cos\phi) + y \sin\phi + 2pe = 0$$

is a tangent to $x^2 + y^2 = e^2(x + 2p)^2$. Find the coordinates of the point of contact.

15. Find the line-equation of a conic of eccentricity e, whose focus the origin, and directrix the line $(q, 0)$.

16. Show that a line-equation of the second degree represents an ellipse if the absolute term and the discriminant have the same sign, a hyperbola if they have opposite signs.

Examples 17–25 relate to the conic $ax^2 + 2hxy + by^2 = 1$.

17. Prove that $ax + hy + m(hx + by) = 0$ is the diameter conjugate to $y = mx$.

18. Prove that the equation of any pair of conjugate diameters can be written $x(hx + by) + \lambda y(ax + hy) = 0$.

19. Prove that $ax^2 - by^2 = 0$, and also $hx^2 + (a + b)xy + hy^2 = 0$, are conjugate diameters.

20. Prove that the slopes of conjugate diameters are connected by the relation $a + h(m_1 + m_2) + bm_1m_2 = 0$.

21. Prove that the slopes of equal diameters are connected by the relation $2h - (a - b)(m_1 + m_2) - 2hm_1m_2 = 0$.

22. Find diameters that shall be both equal and perpendicular.

23. Prove that the equiconjugate diameters are—

$$2h[hx^2 + (a + b)xy + hy^2] + (a - b)(ax^2 - by^2) = 0.$$

24. Prove that the lines $a'x^2 + 2h'xy + b'y^2 = 0$ are conjugate diameters if $ab' + a'b - 2hh' = 0$, and that they are equal diameters if $h'(a - b) = h(a' - b')$.

25. Prove that if $u = 0$ represents one pair of conjugate diameters, and $v = 0$ another pair, then all pairs are given by $\lambda u + v = 0$.

26. Prove that if the asymptotes of any conic are $u = 0$, $v = 0$, then pairs of conjugate diameters are $u \pm \lambda v = 0$.

CHAPTER XII

SYSTEMS OF CONICS

167. IN § 46 it was shown that if u, v are linear expressions in x, y, the line $\lambda u + v = 0$ passes through the point of intersection of the lines $u = 0$, $v = 0$. A similar result holds when u, v are not linear; the curve $\lambda u + v = 0$ passes through all points common to the curves $u = 0$, $v = 0$. For if the coordinates of a point P make $u = 0$ and $v = 0$, they make $\lambda u + v = 0$.

Similarly the curve $\lambda uv + w_1 w_2 = 0$ passes through every point in which either of the curves $u = 0$, $v = 0$ meets either of the curves $w_1 = 0$, $w_2 = 0$.

For example, the equation $\frac{x^2}{18} + \frac{y^2}{8} = 1$ may be written—

$$\frac{x^2 - 9}{18} + \frac{y^2 - 4}{8} = 0,$$

or $$4(x - 3)(x + 3) + 9(y - 2)(y + 2) = 0;$$

this form shows that the ellipse passes through the points in which the lines $x = \pm 3$ meet the lines $y = \pm 2$.

In particular, if u, v are expressions of the second degree, the conic $\lambda u + v = 0$ passes through all the points of intersection of the conics $u = 0$, $v = 0$. Moreover, the conic $\lambda u + v = 0$ meets the conic $u = 0$ (or $v = 0$) at these points only; for if $\lambda u + v = 0$ and $u = 0$, then also $v = 0$.

168. This theorem can be utilised for finding the common points of two conics. The conic $\lambda u + v = 0$, in which λ may have any value, passes through these points. Choose λ so that $\lambda u + v = 0$ may represent straight lines. If

$$u = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$$

and
$$v = a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c',$$

so that

$$\lambda u + v = (\lambda a + a')x^2 + 2(\lambda h + h')xy + (\lambda b + b')y^2 + 2(\lambda g + g')x + 2(\lambda f + f')y + (\lambda c + c'),$$

the condition is (§ 154)—

$$(\lambda a + a')(\lambda b + b')(\lambda c + c') + 2(\lambda f + f')(\lambda g + g')(\lambda h + h') - (\lambda a + a')(\lambda f + f')^2 - (\lambda b + b')(\lambda g + g')^2 - (\lambda c + c')(\lambda h + h')^2 = 0,$$

that is,
$$D\lambda^3 + \ldots + \ldots + D' = 0.$$

Since this is a cubic equation, there are three values for λ. Let λ_1 be any one of these, then $\lambda_1 u + v = 0$ becomes $(lx + my + n)(l'x + m'y + n') = 0$. Each of these two lines meets $u = 0$ in two points, which lie also on $v = 0$; hence the conics $u = 0$, $v = 0$ have *four* common points, real or imaginary, which we can find by solving (i) one cubic equation, (ii) two quadratic equations.

Similarly two equations of the second degree in line-coordinates have four sets of solutions; that is, two conics have four common tangents.

169. Further, the equation of any conic through the four points of intersection of $u = 0$, $v = 0$ can be written $\lambda u + v = 0$.

Since the general equation of a conic, namely,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

involves six coefficients a, b, c, f, g, h, there are five quantities to be determined, the ratios of five of these coefficients to the remaining one. The condition that a conic shall pass through a particular point (x_1, y_1) imposes on these unknown coefficients a single relation,

$$ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0;$$

five such relations determine $a:b:c:f:g:h$ uniquely, by means of five linear equations. Hence one conic passes through any five arbitrary points. In the case considered, four of these are given as the intersections of $u = 0$, $v = 0$, hence the conic can be made to pass through one more point, but it is thereby determined. By means of the undetermined quantity λ in the equation $\lambda u + v = 0$, we can make this conic, which already passes through the four intersections of $u = 0$, $v = 0$, pass through any desired fifth point. Hence $\lambda u + v = 0$ is the equation of any conic through the intersections of $u = 0$, $v = 0$, and for all values of λ the equation represents all these conics. That is, $\lambda u + v = 0$ is the equation of the *system of conics*, or *family of conics*, through the four points of intersection of $u = 0$, $v = 0$.

By a precisely similar argument, if $\phi = 0$, $\psi = 0$ are the equations of two conics in line-coordinates, then $\lambda\phi + \psi = 0$ is the equation of the system of all conics that touch the four common tangents.

The system of conics with four fixed points is called a

pencil of conics; the system with four fixed tangents is a *range* of conics.

170. *Example* i.—The angles made with an axis of a conic by a pair of common chords of the conic and a circle are supplementary.

The equation of any conic can be reduced to—

$$ax^2 + by^2 + 2gx + 2fy + c = 0,$$

by taking the axis of x to be parallel to an axis of the conic (§ 155). The equation of any circle is—

$$x^2 + y^2 + 2g'x + 2f'y + c' = 0.$$

Every conic through the common points has an equation of the form—

$$(a + \lambda)x^2 + (b + \lambda)y^2 + 2(g + \lambda g')x + 2(f + \lambda f')y + c + \lambda c' = 0.$$

If λ is chosen to satisfy the condition $\mathrm{D} = 0$, this represents straight lines; these lines are parallel to $(a + \lambda)x^2 + (b + \lambda)y^2 = 0$, that is, to $y = \pm\sqrt{-\dfrac{a+\lambda}{b+\lambda}}\,.\,x$; and these lines make supplementary angles with the axis of x.

Example ii.—If two central conics have parallel axes, their four common points lie on a circle, and conversely.

If the axis of x is parallel to an axis of one conic, the equation is $ax^2 + by^2 + 2gx + 2fy + c = 0$. Make no assumption as to the axes of the second conic, but take the general equation—

$$a'x' + 2h'xy + b'y^2 + 2g'x + 2f'y + c' = 0.$$

Every conic through the common points of these has an equation—

$$(\lambda a + a')x^2 + 2h'xy + (\lambda b + b')y^2 + 2(\lambda g + g')x + 2(\lambda f + f')y$$
$$+ (\lambda c + c') = 0.$$

This represents a circle if $h' = 0$, $\lambda a + a' = \lambda b + b'$.

The first of these conditions makes the equation of the second conic $a'x^2 + b'y^2 + 2g'x + 2f'y + c' = 0$, and thus shows that an axis is parallel to the axis of x. Hence the conics have parallel axes.

The value of λ given by the second condition determines the equation of the circle through the four points.

EXAMPLES.

1. Prove that if a circle can be passed through the four common points of two parabolas, the axes of the parabolas must be at right angles.

2. Prove that a pencil of conics contains one rectangular hyperbola and two parabolas; but no circle, unless a certain condition is satisfied.

3. How many conics of a pencil have their axes in a specified direction—(i) if the pencil does not contain a circle, (ii) if the pencil does contain a circle?

4. Show that by a proper choice of axes the equation of a pencil of conics can be reduced to the form—

$$ax^2 + by^2 + 2gx + 2fy + c + \lambda(2xy + c') = 0.$$

5. Every conic through the four common points of two rectangular hyperbolas is a rectangular hyperbola.

6. How are the two parabolas accounted for in a pencil of rectangular hyperbolas?

7. Prove that the common chords of two rectangular hyperbolas are at right angles.

8. If a rectangular hyperbola circumscribes a triangle, it passes through the orthocentre.

9. The asymptotes of two conics intersect on a circle; prove that the conics intersect on a concentric circle.

171. If the two conics are circles—

$$u = x^2 + y^2 + 2gx + 2fy + c,$$
$$v = x^2 + y^2 + 2g'x + 2f'y + c',$$

the condition that $\lambda u + v = 0$ represent straight lines is $(\lambda + 1)^2(\lambda c + c') - (\lambda + 1)(\lambda f + f')^2 - (\lambda + 1)(\lambda g + g')^2 = 0.$ One value of λ is -1, the other two values are the roots of the quadratic equation—

$$(\lambda + 1)(\lambda c + c') - (\lambda f + f')^2 - (\lambda g + g')^2 = 0.$$

With the value -1 for λ, the equation $\lambda u + v = 0$ reduces to $2(g - g')x + 2(f - f')y + c - c' = 0$, which

represents, apparently, only one line; this line determines *two* common points, real, or imaginary. This result conflicts with the conclusion of § 168, that two conics have four common points. A more thorough investigation will show, however, that this discrepancy is only apparent.

If the axis of x is taken to pass through the centres of the two circles, $f = 0$ and $f' = 0$; the common chord is then $2(g - g')x + c - c' = 0$, a line perpendicular to the axis of x. Take it as axis of y; then $c = c'$. The two circles are—

$$u = x^2 + y^2 + 2gx + c = 0,$$
$$v = x^2 + y^2 + 2g'x + c = 0.$$

The equation determining the values of λ, other than -1, for which $\lambda u + v = 0$ represents straight lines, is now $c(\lambda + 1)^2 - (\lambda g + g')^2 = 0$. Hence—

$$\lambda = -\frac{g' - \sqrt{c}}{g - \sqrt{c}}, \text{ or } -\frac{g' + \sqrt{c}}{g + \sqrt{c}}.$$

The two pairs of common chords are therefore—

$$(g' - \sqrt{c})u - (g - \sqrt{c})v = 0,$$

and $$(g' + \sqrt{c})u - (g + \sqrt{c})v = 0,$$

that is, after rejection of the factor $g' - g$ which presents itself—

$$x^2 + y^2 + 2\sqrt{c}\,x + c = 0,$$

and $$x^2 + y^2 - 2\sqrt{c}\,x + c = 0,$$

which may be written—

$$(x + \sqrt{c})^2 + y^2 = 0,$$

and $$(x - \sqrt{c})^2 + y^2 = 0.$$

One pair of common chords is therefore—

$$x + \sqrt{c} \pm iy = 0,$$

and another pair is $$x - \sqrt{c} \pm iy = 0;$$

hence the circles intersect in four points, as follows—

(1) $x + \sqrt{c} + iy = 0$ meets $x - \sqrt{c} - iy = 0$ at $(0, \sqrt{-c})$,
(2) $x + \sqrt{c} - iy = 0$ meets $x - \sqrt{c} + iy = 0$ at $(0, -\sqrt{-c})$,
(3) $x + \sqrt{c} + iy = 0$ meets $x - \sqrt{c} + iy = 0$ at infinity,
(4) $x + \sqrt{c} - iy = 0$ meets $x - \sqrt{c} - iy = 0$ at infinity.

Of these four points, the first two lie on the real common chord, $u - v = 0$; the other two lie at infinity, in directions given by $y = \pm ix$, that is, in particular imaginary directions.

Since these directions do not depend on the particular circles, we arrive at the conclusion that all circles pass through the same two imaginary points at infinity. These are called the *circular points* of the plane.

Note.—The same conclusion results from the equation of the asymptotes. For the circles—

$$(x - p)^2 + (y - q)^2 = r^2,$$
$$(x - p')^2 + (y - q')^2 = r'^2,$$
$$(x - p'')^2 + (y - q'')^2 = r''^2,$$

these are

$$(x - p)^2 + (y - q)^2 = 0,$$
$$(x - p')^2 + (y - q')^2 = 0,$$
$$(x - p'')^2 + (y - q'')^2 = 0,$$

that is,

$$y - q = \pm i(x - p),$$
$$y - q' = \pm i(x - p'),$$
$$y - q'' = \pm i(x - p'');$$

hence they are parallel, which shows that the circles have two points in common at infinity.

Example.—Prove that every conic through the circular points is a circle.

172. The lines that join any point (p, q) to the circular points are $y - q = \pm i(x - p)$. These are called the *isotropic lines*, or the *null lines* through the point. The first name refers to the fact that the expression for the tangent of the angle made by an isotropic line with a line $y = mx$ is the same for all values of m; for it is $\dfrac{i - m}{1 + im}$ (§ 33), that is, $i\dfrac{i - m}{i - m}$, or i. The second name is given because the distance between any two points on such a line is zero. To prove this, take two points (x_1, y_1), (x_2, y_2) on an isotropic line through the origin $y = ix$; their distance apart is—

$$\begin{aligned} d &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(x_1 - x_2)^2 + (ix_1 - ix_2)^2} \\ &= \sqrt{(x_1 - x_2)^2 - (x_1 - x_2)^2} = 0. \end{aligned}$$

The equation of a pair of isotropic lines satisfies the analytical conditions for a circle, namely, $a = b$ and $h = 0$. For the equations of the two lines have been shown to be $y - q = \pm i(x - p)$, hence their combined equation is $(x - p)^2 + (y - q)^2 = 0$.

Note.—From one point of view, this may be looked upon as representing a circle of zero radius, a point-circle.

The isotropic line $x - p + i(y - q) = 0$ has the coordinates $\xi = \dfrac{-1}{p + iq}$, $\eta = \dfrac{-i}{p + iq}$, hence for every such line $\xi + i\eta = 0$; this is the equation of one circular point. For the isotropic line $x - p - i(y - q) = 0$, $\xi = \dfrac{-1}{p - iq}$,

$\eta = \dfrac{+\,i}{p - iq}$, hence for every such line $\xi - i\eta = 0$; this is the equation of the other circular point. The equation of the pair of points is therefore—

$$(\xi + i\eta)(\xi - i\eta) = 0,$$

that is,

$$\xi^2 + \eta^2 = 0.$$

173. If $u = 0$, $v = 0$ are two circles, then for all values of λ the equation $\lambda u + v = 0$ represents a circle (or a pair of isotropic lines); for the equation is—

$$(\lambda + 1)x^2 + (\lambda + 1)y^2 + 2(\lambda g + g')x + 2(\lambda f + f')y + \lambda c + c' = 0.$$

Hence if two conics of a pencil are circles, then all conics of the pencil are circles. This is because two of the four points by which the pencil is determined are the circular points, and any conic through these is necessarily a circle (Ex. § 171.) Whether the other two intersections of the circles are real or imaginary, the line that joins them, $u - v = 0$, is real. This line is called the radical axis of the circles (§ 147); the system of circles $\lambda u + v = 0$ is called a coaxal system. A coaxal system of circles is simply a pencil of circles. The equations of two circles of the system can be written (§ 171)—

$$x^2 + y^2 + 2gx + c = 0,$$
$$x^2 + y^2 + 2g'x + c = 0;$$

the equation of any circle of the system is therefore—

$$(\lambda + 1)x^2 + (\lambda + 1)y^2 + 2(g\lambda + g')x + (\lambda + 1)c = 0,$$

that is,

$$x^2 + y^2 + 2\frac{g\lambda + g'}{\lambda + 1}x + c = 0.$$

This may be expressed in the form—If g is a variable parameter, the equation $x^2 + y^2 + 2gx + c = 0$ represents a coaxal system of circles.

The common points of these circles are $(0, \pm\sqrt{-c})$, hence they are real or imaginary according as c is negative or positive.

The equation of a circle of the system is—

$$(x + g)^2 + y^2 = g^2 - c.$$

The centre is $(-g, 0)$, the radius is $\sqrt{g^2 - c}$. Hence the centres of all circles of the system lie on a straight line, perpendicular to the radical axis.

The system contains two circles of zero radius, or point circles; these are given by $g^2 = c$. Their equations are—

$$(x \pm \sqrt{c})^2 + y^2 = 0.$$

The points thus determined, $(\pm\sqrt{c}, 0)$, are called the limiting points of the coaxal system. They are real if c is positive, that is, if the common points of the circles are imaginary; but if the common points are real, the limiting points are imaginary.

Note.—The point-circles of the system are the same as the pairs of imaginary common chords.

EXAMPLES.

1. Prove that the polar of a limiting point with respect to all circles of the system is the same, and that it passes through the other limiting point.

2 Prove that if the common points of the system are imaginary, the circles fall into two sets, each set having one limiting point inside all its circles.

3. Prove that any circle which passes through the limiting points is orthogonal to all the circles of the system (§ 147).

4. Prove that the three radical axes of any three circles (not coaxal) taken in pairs meet at one point (the radical centre).

5. Prove that any circle that is orthogonal to three given circles $u = 0$, $v = 0$, $w = 0$ is orthogonal to the circle $\lambda u + \mu v + \nu w = 0$.

174. If u, v, w_1, w_2 are linear expressions, the equation $\lambda uv + ww' = 0$ represents a conic which passes through the four points P, P′, Q, Q′, in which the lines $u = 0$, $v = 0$ meet the lines $w = 0$, $w' = 0$ (Fig. 92). If the

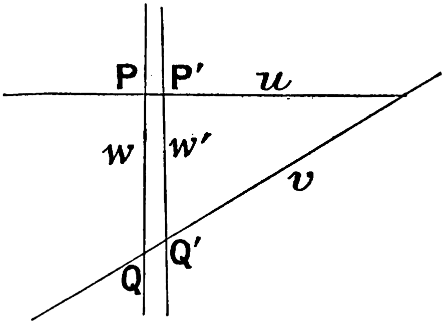

Fig. 92.

lines w, w' are very close together, the points P, P′ and also the points Q, Q′ are very close together; if the lines are indistinguishable, the points P, P′ are indistinguishable, as also Q, Q′, hence the lines u, v are tangents. That is, the equation $\lambda uv + w^2 = 0$ represents a conic, which has the lines $u = 0$, $v = 0$ as tangents, with $w = 0$ as chord of contact.

For example, the equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ can be written in the form $\dfrac{b^2}{a^2}(x^2 - a^2) + y^2 = 0$, that is, $\dfrac{b^2}{a^2}(x - a)(x + a) + y^2 = 0$. This shows that the lines $x - a = 0$, $x + a = 0$ are tangents, whose points of contact lie on $y = 0$.

Similarly if $u = 0$ is a conic, met by the line $w = 0$ at P, Q, and by the line $w' = 0$ at P′, Q′, the equation $\lambda u + ww' = 0$ represents a conic through the points P, P′, Q, Q′; and if w, w' are indistinguishable, this conic, $\lambda u + w^2 = 0$, touches $u = 0$ at the points P, Q. Hence if λ has the value which makes this conic break up into straight lines, they are the tangents to $u = 0$ at the points where it is met by $w = 0$. That

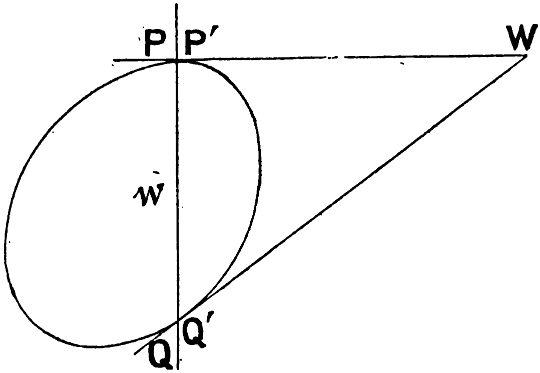

Fig. 93.

is, the tangents to the conic $u = 0$ at points on the line $w = 0$ are given by the equation $\lambda u + w^2 = 0$, where λ is properly chosen.

For example, the tangents to $x^2 + y^2 - 25 = 0$ where it is met by $x + 7y - 25 = 0$ are given by the equation—

$$\lambda(x^2 + y^2 - 25) + (x + 7y - 25)^2 = 0,$$

where λ satisfies the condition for straight lines. This reduces to $\lambda = -25$; hence the tangents are—

$$25(x^2 + y^2 - 25) - (x + 7y - 25)^2 = 0,$$

that is, $12x^2 - 7xy - 12y^2 + 25x + 175y - 625 = 0,$

or $$(4x + 3y - 25)(3x - 4y + 25) = 0.$$

A general expression can however be found for the value of λ. If W is the pole of the line $w = 0$, it suffices to make the conic $\lambda u + w^2 = 0$ pass through W (Fig. 93). For this conic already passes through P, P′, hence if we make it go through W, it has three points on the line PW, which must therefore form a part of the "conic." Similarly on account of the three points Q, Q′, W on the tangent at Q, this line forms a part. Hence to determine λ, find the pole of $w = 0$, and express that the conic $\lambda u + w^2 = 0$ passes through this point. If the pole is (x', y'), the condition is $\lambda u' + w'^2 = 0$; hence $\lambda = -\dfrac{w'^2}{u'}$. Now if $u = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$, the polar of (x', y') is $w = 0$, where

$$w = axx' + hxy' + hx'y + byy' + g(x + x') + f(y + y') + c.$$

Hence $w' = ax'^2 + 2hx'y' + by'^2 + 2gx' + 2fy' + c$, which is the same as u'. Consequently $\lambda = -\dfrac{w'^2}{u'} = -u'$, and the equation of the pair of tangents becomes—

$$uu' - w^2 = 0.$$

If (x', y') is given, then w is known, for $w = 0$ is the polar of (x', y'); hence this equation gives also the pair of tangents from a point (x', y').

For example, the pole of $x + 7y - 25 = 0$ with respect to $x^2 + y^2 - 25 = 0$ is the point $(1, 7)$; hence $u' = 1 + 49 - 25 = 25$, and the tangents (already found by a longer process) are $25u = w^2$, that is, $25(x^2 + y^2 - 25) = (x + 7y - 25)^2$.

Example.—Find the locus of the point of intersection of perpendicular tangents to $u = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

The tangents from (x', y') are—

$$uu' = [x(ax' + hy' + g) + y(hx' + by' + f) + gx' + fy' + c]^2.$$

These are perpendicular if the coefficient of x^2 + the coefficient of $y^2 = 0$; that is, if

$$au' - (ax' + hy' + g)^2 + bu' - (hx' + by' + f)^2 = 0;$$

hence the locus of (x', y') is—

$$(a + b)(ax^2 + 2hxy + by^2 + 2gx + 2fy + c) - (ax + hy + g)^2 - (hx + by + f)^2 = 0,$$

that is—

$$(ab - h^2)(x^2 + y^2) - 2(hf - bg)x - 2(gh - af)y + bc - f^2 + ca - g^2 = 0.$$

With the notation of § 165 this is—

$$C(x^2 + y^2) - 2Gx - 2Fy + A + B = 0.$$

This represents a circle unless the conic is a parabola ($C = 0$), and then it represents a straight line, the directrix of the parabola.

Examples.

1. Apply the equation of the tangents from a point to find the tangents from the centre (the asymptotes).

2. Show that the tangents parallel to the axis of x are $Cy^2 - 2Fy + B = 0$, and that the tangents parallel to the axis of y are $Cx^2 - 2Gx + A = 0$. Show also that the points of contact of these pairs of tangents lie on the lines—

$$ax + hy + g = 0, \quad hx + by + f = 0.$$

175. If the origin is taken at a focus of a conic, and the axis of y parallel to the directrix, the equation of the directrix is $x - k = 0$, and the equation of the conic is—

$$\sqrt{x^2 + y^2} = \pm\, e(x - k),$$

that is,

$$x^2 + y^2 = e^2(x - k)^2.$$

This may be written—

$$(x + iy)(x - iy) = e^2(x - k)^2,$$

a form which shows that the lines $x + iy = 0$, $x - iy = 0$ are tangents, for which $x - k = 0$ is the chord of contact.

These lines, $x^2 + y^2 = 0$, are the isotropic lines through the focus; hence the tangents from a focus satisfy the analytical conditions for a circle.

Moreover, if the tangents from a point are isotropic lines, the point is a focus. To prove this, take the point as origin, and let the polar of O be $lx + my + n = 0$; then since $x + iy = 0$, $x - iy = 0$ are tangents with $lx + my + n = 0$ as chord of contact, the equation of the conic is—

$$x^2 + y^2 = \lambda . (lx + my + n)^2.$$

Now if P is a point (x, y), $OP^2 = x^2 + y^2$, and the distance from P to the line $lx + my + n = 0$ is—

$$MP = \pm \frac{lx + my + n}{\sqrt{l^2 + m^2}};$$

hence the equation of the conic expresses that—

$$OP^2 = \lambda(l^2 + m^2) . MP^2,$$

or that OP is in a constant ratio to MP. Hence O is a focus of the conic.

176. A focus of a conic is therefore the point of intersection of two isotropic tangents; and to find a focus, (x', y'), write down the equation for the pair of tangents from (x', y'), and express that this satisfies the analytical conditions for a circle.

The equation of the tangents from (x', y') to

$$u = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

is $\quad uu' = w^2,$

where $\quad w = x(ax' + hy' + g) + y(hx' + by' + f) + gx' + fy' + c.$

If this is written—

$$a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c' = 0,$$

the conditions to be satisfied are $a' - b' = 0$, $h' = 0$.

$$\begin{aligned} \text{Now } a' &= au' - (ax' + hy' + g)^2 \\ &= a(ax'^2 + 2hx'y' + by'^2 + 2gx' + 2fy' + c) \\ &\qquad - (ax' + hy' + g)^2 \\ &= (ab - h^2)y'^2 - 2(gh - af)y' + (ca - g^2) \\ &= Cy'^2 - 2Fy' + B; \\ b' &= bu' - (hx' + by' + f)^2 \\ &= Cx'^2 - 2Gx' + A; \\ h' &= hu' - (ax' + hy' + g)(hx' + by' + f) \\ &= -(ab - h^2)x'y' + (gh - af)x' + (hf - bg)y' \\ &\qquad - (fg - ch) \\ &= -Cx'y' + Fx' + Gy' - H. \end{aligned}$$

Hence a focus is any point common to the two curves—

$$C(x^2 - y^2) - 2Gx + 2Fy + A - B = 0,$$
$$Cxy - Fx - Gy + H = 0,$$

both of which are rectangular hyperbolas. There are consequently four foci, which are the common points of a pencil of rectangular hyperbolas. If however the conic $u = 0$ is a parabola, $C = 0$, and the one focus is found as the point of intersection of the lines—

$$2Gx - 2Fy - A + B = 0,$$
$$Fx + Gy - H = 0.$$

If the axes of a central conic $u = 0$ are axes of co-ordinates, so that the equation is $\frac{x^2}{\alpha} + \frac{y^2}{\beta} = 1$, the two hyperbolas become—

$$x^2 - y^2 = \alpha - \beta, \; xy = 0;$$

the pencil of focal hyperbolas is—

$$x^2 + 2\lambda xy - y^2 - \alpha + \beta = 0.$$

The four foci lie, two on the transverse axis $y = 0$, at the points $x = \pm\sqrt{\alpha - \beta}$, these are real; and two on the conjugate axis $x = 0$, at the points $y = \pm\sqrt{\beta - \alpha}$, these are imaginary.

177. Since the foci lie on the axes, one of the rectangular hyperbolas through the foci (the focal hyperbolas),

$$\begin{gathered} C(x^2 - y^2) - 2Gx + 2Fy + A - B \\ + \lambda(Cxy - Fx - Gy + H) = 0 \end{gathered}$$

represents the axes. We already know (§ 164) that the axes are parallel to $\dfrac{x^2 - y^2}{a - b} - \dfrac{xy}{h} = 0$; hence $\lambda = -\dfrac{a - b}{h}$, and the equation of the axes is therefore—

$$\begin{gathered} Ch(x^2 - y^2) - 2Ghx + 2Fhy + Ah - Bh - C(a - b)xy \\ + F(a - b)x + G(a - b)y - H(a - b) = 0. \end{gathered}$$

Example.—Find the foci and axes of—

$$5x^2 + 4xy + 2y^2 + 12y + 6 = 0.$$

For this equation—

$$\begin{aligned} A &= 12 - 36 = -24, & F &= 0 - 30 = -30, \\ B &= 30 - 0 = 30, & G &= 12 - 0 = 12, \\ C &= 10 - 4 = 6, & H &= 0 - 12 = -12. \end{aligned}$$

Dropping the common factor -6, we find for the line-equation—

$$4\xi^2 + 4\xi\eta - 5\eta^2 - 4\xi + 10\eta - 1 = 0.$$

The two hyperbolas are $x^2 - y^2 - 4x - 10y - 9 = 0$.

$$xy + 5x - 2y - 2 = 0.$$

The axes are—

$$2(x^2 - y^2 - 4x - 10y - 9) - 3(xy + 5x - 2y - 2) = 0,$$

that is, $$2x^2 - 3xy - 2y^2 - 23x - 14y - 12 = 0,$$

or $$(2x + y + 1)(x - 2y - 12) = 0.$$

The axis $2x + y + 1$ meets $xy + 5x - 2y - 2 = 0,$

where $$-x(2x + 1) + 5x + 4x + 2 - 2 = 0,$$

that is, $$-2x^2 + 8x = 0, x = 0 \text{ or } 4;$$

hence the two foci that lie on $2x + y + 1 = 0$ are $(0, -1)$, $(4, -9)$. Since these are real, this is the transverse axis, and $x - 2y - 12 = 0$ is the conjugate axis. The centre is $(2, -5)$.

Note.—It is not customary to find the imaginary foci.

The directrices are most simply found as the polars of the foci. The polar of (x', y') is—

$$x(5x' + 2y') + y(2x' + 2y' + 6) + (6y' + 6) = 0;$$

hence the directrix that corresponds to $(0, -1)$ is $-2x + 4y = 0$, or $x - 2y = 0$; and the other directrix is $2x - 4y - 48 = 0$, or $x - 2y - 24 = 0$.

EXAMPLES.

1. Find by the process of §§ 176, 177 the foci, axes, vertices, and (when real) the asymptotes of—

(i) $7x^2 - 8xy + y^2 + 10x + 2y - 44 = 0.$
(ii) $9x^2 - 6xy + y^2 - 40x - 20y + 75 = 0.$
(iii) $65x^2 + 24xy + 20y^2 - 178x - 104y + 125 = 0.$
(iv) $160x^2 - 18xy + 160y^2 - 906x - 906y - 151^2 = 0.$
(v) $11x^2 - 8xy + 11y^2 - 96x + 54y + 114 = 0.$
(vi) $5x^2 - 24xy - 5y^2 - 280x - 4y + 62 = 0.$

2. Draw the conics (i)–(vi) in Ex. 1 without reducing the equation.

3. Prove that any conic through the four foci of a conic is a rectangular hyperbola.

4. Prove that the vertices of a rectangle circumscribed to a conic lie on a focal hyperbola, namely, on that one whose axes are parallel to the sides of the rectangle.

5. Find the locus of the vertices of the focal hyperbolas of $\dfrac{x^2}{\alpha} + \dfrac{y^2}{\beta} = 1$. Find also the locus of the foci.

178. Conics with the same foci, *confocal conics*, form a system of the kind mentioned in § 169; they have four fixed tangents, two from each circular point, hence they form a range. This particular range is important, and has interesting geometrical properties.

Since the axes are the same in position for all such conics, the equations of the conics can all be written in the form—

$$\frac{x^2}{p}+\frac{y^2}{q}=1;$$

and since the foci are the same, $p - q$ has the same value for all the conics of the system. Hence if for one conic $p = \alpha$ and $q = \beta$, then for any other conic $p = \alpha + \lambda$, $q = \beta + \lambda$. The equation of the system is therefore—

$$\frac{x^2}{\alpha+\lambda}+\frac{y^2}{\beta+\lambda}=1,$$

where λ assumes all values. This can also be written—

$$\frac{x^2}{k}+\frac{y^2}{k-c^2}=1,$$

where k is the parameter.

The line-equation of the system of confocals is therefore—

$$(\alpha+\lambda)\xi^2+(\beta+\lambda)\eta^2=1,$$

or

$$k\xi^2+(k-c^2)\eta^2=1,$$

according to the form used.

For the present, the first form only will be used; the system of confocals is expressed by either of the equations—

$$(\beta+\lambda)x^2+(\alpha+\lambda)y^2=(\alpha+\lambda)(\beta+\lambda),$$
$$(\alpha+\lambda)\xi^2+(\beta+\lambda)\eta^2=1.$$

Notice that the parameter λ enters linearly in the line-equation, but in quadratic form in the point-equation.

179. As regards any system of curves, it is usually of importance to know how many curves pass through an arbitrary point, how many touch an arbitrary line. A curve of this system passes through (x', y') if

$$(\beta + \lambda)x'^2 + (\alpha + \lambda)y'^2 = (\alpha + \lambda)(\beta + \lambda).$$

This is a quadratic equation for λ, consequently there are two values; two curves of the system pass through an arbitrary point. That these are necessarily real, and are an ellipse and hyperbola, is at once evident by geometrical considerations. An ellipse can be described with given foci S, S′, through any point P, by taking for major axis the sum of the two distances SP, S′P; and a hyperbola by taking for major axis the difference of these two distances. Moreover, by §§ 144, 146, vi, the tangents to the two curves at P are the bisectors of the exterior and interior angles SPS′, hence the curves are orthogonal.

Algebraically, the curves are proved to be orthogonal as follows. The tangents to—

$$\frac{x^2}{\alpha + \lambda} + \frac{y^2}{\beta + \lambda} = 1,$$

$$\frac{x^2}{\alpha + \lambda'} + \frac{y^2}{\beta + \lambda'} = 1,$$

at a point (x_1, y_1) are—

$$\frac{xx_1}{\alpha + \lambda} + \frac{yy_1}{\beta + \lambda} = 1,$$

$$\frac{xx_1}{\alpha + \lambda'} + \frac{yy_1}{\beta + \lambda'} = 1;$$

these are perpendicular if

$$\frac{x_1^2}{(a+\lambda)(a+\lambda')}+\frac{y_1^2}{(\beta+\lambda)(\beta+\lambda')}=0.$$

Now since (x_1, y_1) is on each of the conics,

$$\frac{x_1^2}{a+\lambda}+\frac{y_1^2}{\beta+\lambda}=1$$

and

$$\frac{x_1^2}{a+\lambda'}+\frac{y_1^2}{\beta+\lambda'}=1;$$

from these, by subtraction,

$$\frac{(\lambda'-\lambda)x_1^2}{(a+\lambda)(a+\lambda')}+\frac{(\lambda'-\lambda)y_1^2}{(\beta+\lambda)(\beta+\lambda')}=0,$$

that is—

$$\frac{x_1^2}{(a+\lambda)(a+\lambda')}+\frac{y_1^2}{(\beta+\lambda)(\beta+\lambda')}=0.$$

Hence the condition of perpendicularity is satisfied.

Only one conic touches an arbitrary line (ξ', η'); for λ must satisfy the equation—

$$(a+\lambda)\xi'^2+(\beta+\lambda)\eta'^2=1,$$

which is of the first degree. From §§ 144, 146, vi, it appears that the conic is a hyperbola if the line separates the two foci, an ellipse if it does not.

180. Any conic of the system is known when the value of λ is known. Since the semi-axes a, b are $\sqrt{a+\lambda}$ and $\sqrt{\beta+\lambda}$ for an ellipse, $\sqrt{a+\lambda}$ and $\sqrt{-(\beta+\lambda)}$ for a hyperbola, if λ is known the semi-axes are known, and conversely. And since two conics pass through a point of the plane, every point is determined by the semi-axes of

the two confocals that pass through it, provided that the quadrant in which the point lies is known, for two confocals meet in four points, symmetrically placed in the four quadrants (Fig. 94).

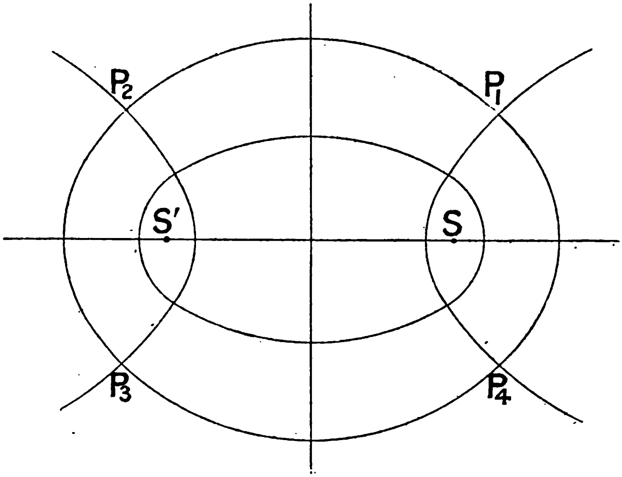

Fig. 94.

The Cartesian coordinates of any point are very simply expressed in terms of the semi-axes of the confocals through the point. We have—

$$\frac{x^2}{a+\lambda}+\frac{y^2}{\beta+\lambda}=1,$$

$$\frac{x^2}{a+\lambda'}+\frac{y^2}{\beta+\lambda'}=1,$$

hence $$x^2\left[\frac{\beta+\lambda}{a+\lambda}-\frac{\beta+\lambda'}{a+\lambda'}\right]=\lambda-\lambda',$$

that is, $$x^2\frac{(a-\beta)(\lambda-\lambda')}{(a+\lambda)(a+\lambda')}=\lambda-\lambda',$$

therefore $$x^2 = \frac{(\alpha + \lambda)(\alpha + \lambda')}{\alpha - \beta};$$

and $$y^2\left[\frac{\alpha + \lambda}{\beta + \lambda} - \frac{\alpha + \lambda'}{\beta + \lambda'}\right] = \lambda - \lambda',$$

therefore $$y^2 = -\frac{(\beta + \lambda)(\beta + \lambda')}{(\alpha - \beta)}.$$

The value of y^2 shows that if $\beta + \lambda$ is positive, then $\beta + \lambda'$ must be negative for real points of intersection. Now let the semi-axes of the ellipse be a_1, b_1, so that $\alpha + \lambda = a_1^2$, $\beta + \lambda = b_1^2$; and the semi-axes of the hyperbola a_2, b_2, so that $\alpha + \lambda' = a_2^2$, $\beta + \lambda' = -b_2^2$; also write c^2 for $\alpha - \beta$. Then $x^2 = \frac{a_1^2a_2^2}{c^2}$, $y^2 = \frac{b_1^2b_2^2}{c^2}$; the coordinates of a point, expressed in terms of the semi-axes of the two confocals through the point, are—

$$\left(\pm\frac{a_1a_2}{c}, \pm\frac{b_1b_2}{c}\right).$$

181. Since the pencil of focal hyperbolas for the general conic is $C(x^2 - y^2) - 2Gx + 2Fy + A - B + \lambda[Cxy - Fx - Gy + H] = 0$, two conics are confocal if their line-equations differ only by a multiple of $\xi^2 + \eta^2$. For multiply the second line-equation

$$A'\xi^2 + 2H'\xi\eta + B'\eta^2 + 2G'\xi + 2F'\eta + C' = 0$$

by $\frac{C}{C'}$; it then becomes—

$$A''\xi + 2H''\xi\eta + B''\eta^2 + 2G''\xi + 2F''\eta + C = 0.$$

The pencil of focal hyperbolas is—

$$C(x^2 - y^2) - 2G''x + 2F''y + A'' - B''$$
$$+ \lambda[Cxy - F''x - G''y + H''] = 0.$$

This is the same as the pencil of focal hyperbolas for the first conic if $F'' = F$, $G'' = G$, $H'' = H$, $A'' - B'' = A - B$. Hence if $A'' = A + \mu$, then also $B'' = B + \mu$; the line-equations of the two conics are—

$$A\xi^2 + 2H\xi\eta + B\eta^2 + 2G\xi + 2F\eta + C = 0,$$
$$\mu(\xi^2 + \eta^2) + A\xi^2 + 2H\xi\eta + B\eta^2 + 2G\xi + 2F\eta + C = 0.$$

That is, if $\phi = 0$ is the line-equation of any conic, then the line-equation of the system of confocals is $\mu(\xi^2 + \eta^2) + \phi = 0$.

This agrees with the fact that the tangents from the foci are isotropic; the conics have four fixed tangents, which satisfy the equation $\xi^2 + \eta^2 = 0$, and also $\phi = 0$. Hence the conics of the system are $\mu(\xi^2 + \eta^2) + \phi = 0$.

182. Other systems of conics may be mentioned.

(i) If the axes of two conics lie along the same straight lines, and have their lengths in the same ratio, the conics are said to be similar and similarly situated concentric conics. Since $\alpha' : \beta' = \alpha : \beta$, write $\alpha' = k\alpha$, then $\beta' = k\beta$. Hence the equation of the system is—

$$\frac{x^2}{k\alpha} + \frac{y^2}{k\beta} = 1,$$

or

$$\frac{x^2}{\alpha} + \frac{y^2}{\beta} = k \text{ (§ 130).}$$

Such conics have the same asymptotes, and meet only at infinity. The system is of a very simple character; one conic passes through any point, one touches any line.

(ii) Again, conics whose axes lie along the same straight lines, and which have the same director circle, are expressed by the equation—

$$\frac{x^2}{\alpha'} + \frac{y^2}{\beta'} = 1, \text{ where } \alpha' + \beta' = \text{const.}$$

These conics may be called co-directoral.

If α, β are values of α', β' for one conic, the point-equation of the system is—

$$\frac{x^2}{\alpha + \lambda} + \frac{y^2}{\beta - \lambda} = 1;$$

the line-equation is $(\alpha + \lambda)\xi^2 + (\beta - \lambda)\eta^2 = 1$.

Hence two conics of the system pass through any point; one conic touches any line.

The form of the line-equation shows that the conics of the system have four fixed tangents. This equation is—

$$\alpha\xi^2 + \beta\eta^2 - 1 + \lambda(\xi^2 - \eta^2) = 0,$$

hence for all values of λ the conic touches the lines whose coordinates satisfy—

$$\alpha\xi^2 + \beta\eta^2 - 1 = 0,$$
$$\xi^2 - \eta^2 = 0,$$

that is, the lines $\left(\pm\dfrac{1}{\sqrt{\alpha+\beta}},\ \pm\dfrac{1}{\sqrt{\alpha+\beta}}\right)$. Hence the conics have the four fixed tangents—

$$\pm x \pm y + \sqrt{\alpha+\beta} = 0.$$

183. When the equation of a system of conics involves a single parameter, the locus of a point, or the envelope of a line, which has a definite relation to a curve of the system, is found by writing down the equations which connect the coordinates of the point or line with the value of the parameter, and then eliminating the parameter.

Example i.—Find the locus of the centres of conics of the system $\lambda u + v = 0$, where u, v are general conics—

$$u = ax^2 + 2hxy + by^2 + 2gx + 2fy + c,$$
$$v = a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c'.$$

The equations which give the centre are—

$$(\lambda a + a')x + (\lambda h + h')y + \lambda g + g' = 0,$$
$$(\lambda h + h')x + (\lambda b + b')y + \lambda f + f' = 0.$$

From these, eliminate λ. The equations are—

$$\lambda(ax + hy + g) + a'x + h'y + g' = 0,$$
$$\lambda(hx + by + f) + h'x + b'y + f' = 0;$$

hence
$$\frac{ax + hy + g}{a'x + h'y + g'} = \frac{hx + by + f}{h'x + b'y + f'},$$

and the centre-locus of the pencil is therefore the conic—

$$\begin{aligned}(ah' - a'h)x^2 + (ab' - a'b)xy + (b'h - bh')y^2 \\ + (gh' - g'h + af' - a'f)x \\ + (hf' - h'f + b'g - bg')y + gf' - g'f = 0.\end{aligned}$$

Example ii.—Find the locus of the points of contact of tangents from a fixed point (h, k) to a system of concentric similar and similarly situated conics.

Any conic of the system is—

$$\frac{x^2}{\lambda\alpha} + \frac{y^2}{\lambda\beta} = 1.$$

The tangent at x_1, y_1 is
$$\frac{xx_1}{\lambda\alpha} + \frac{yy_1}{\lambda\beta} = 1.$$

This passes through (h, k) if

$$\frac{hx_1}{\lambda\alpha} + \frac{ky_1}{\lambda\beta} = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Also (x_1, y_1) is on the conic, hence—

$$\frac{x_1^2}{\lambda\alpha} + \frac{y_1^2}{\lambda\beta} = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From the equations (1) and (2), eliminate λ. These equations are—

$$\frac{hx_1}{\alpha} + \frac{ky_1}{\beta} = \lambda,$$
$$\frac{x_1^2}{\alpha} + \frac{y_1^2}{\beta} = \lambda,$$

hence
$$\frac{x_1^2}{\alpha} + \frac{y_1^2}{\beta} - \frac{hx_1}{\alpha} - \frac{ky_1}{\beta} = 0;$$

the locus of the point (x_1, y_1) is the conic—

$$\frac{x^2}{\alpha} + \frac{y^2}{\beta} - \frac{hx}{\alpha} - \frac{ky_1}{\beta} = 0.$$

Since this equation can be written—

$$\frac{\left(x-\frac{h}{2}\right)^2}{\alpha}+\frac{\left(y-\frac{k}{2}\right)^2}{\beta}=\frac{h^2}{4\alpha}+\frac{k^2}{4\beta},$$

if m is written for $\frac{h^2}{4\alpha}+\frac{k^2}{4\beta}$, and the origin changed to $\left(\frac{h}{2}, \frac{k}{2}\right)$, the equation becomes $\frac{x^2}{m\alpha}+\frac{y^2}{m\beta}=1$. Hence the locus is a conic, similar to the one given, through the fixed point and the centre of the given conic, with its axes parallel to the axes of the given conic.

Examples.

1. Find the locus of the points of contact of parallel tangents to a system of—

(i) Confocal conics.
(ii) Concentric, similar, similarly situated conics.
(iii) Co-directoral conics.
(iv) Coaxal circles.

2. Find the locus of the points of contact of tangents from a fixed point to each of the four systems mentioned in Ex. 1.

3. Find the locus of the pole of a fixed line with respect to each of the four systems (Ex. 1).

4. Find the envelope of tangents at the points of intersection of a straight line with each of the systems (Ex. 1).

5. Find the envelope of normals at the points of intersection of a straight line with each of the systems (Ex. 1).

6. Find the locus of points, if the normals pass through a fixed point (for each of the systems, Ex. 1).

7. Prove that the locus of the point of intersection of perpendicular tangents to two confocals is a circle through their common points.

8. Prove that the difference of the squares of the distances from the centre to two parallel tangents to two confocal conics is constant.

9. Find the centre-locus for a pencil of rectangular hyperbolas (not concentric).

10. Find the centre-locus for a pencil of conics that contains a circle.

11. Two rectangular hyperbolas have the axes of one as asymptotes of the other. Prove that they cut orthogonally.

12. Two systems of rectangular hyperbolas have fixed asymptotes, the same centre. Prove that one hyperbola of each system passes through any point in the plane, and that these cut at a constant angle, equal to twice the angle made by the asymptotes of one with the asymptotes of the other.

13. Prove that the following systems of curves are orthogonal—

$$\text{(i) } y^2 = \lambda x,\ 2x^2 + y^2 = \mu.$$
$$\text{(ii) } y^3 = \lambda x,\ 3x^2 + y^2 = \mu.$$
$$\text{(iii) } y^3 = \lambda x^2,\ 3x^2 + 2y^2 = \mu.$$
$$\text{(iv) } x^2y = \lambda,\ x^2 - 2y^2 = \mu.$$

14. Find the locus of a point whose polars with respect to the conics—

$$u = ax^2 + by^2 + 2gx + 2fy + c = 0,$$
$$v = 2xy + c' = 0,$$

are parallel. Show that the polars with respect to all conics of the system $u + \lambda v = 0$ are parallel.

15. Find the locus of a point whose polars with respect to the conics $u = 0$, $v = 0$ (Ex. 14) are perpendicular.

16. Prove that the locus of a point whose polars with respect to a circle and a rectangular hyperbola make a constant angle is a rectangular hyperbola.

17. Find the locus of a point whose polars with respect to two rectangular hyperbolas make a constant angle.

18. Two of the common points of a circle and rectangular hyperbola are the extremities of a diameter of the hyperbola; prove that the other two are the extremities of a diameter of the circle, and conversely.

19. Prove that circles described on parallel chords of a rectangular hyperbola as diameters form a coaxal system, whose axis is parallel to the other common chords of the circles and the rectangular hyperbola. Prove that the limiting points lie on the hyperbola.

20. Find the centre-locus of a range of conics.

21. Two pencils of conics each contain a circle, and the asymptotes of the two rectangular hyperbolas are parallel. Find the locus of the common points of conics of the two pencils whose eccentricities are equal.

22. Prove that the axes of the centre-locus of a pencil of conics

are parallel to the asymptotes of the rectangular hyperbola included in the pencil.

23. A parabola is inscribed to a triangle; find the locus of the focus. Prove that the directrix passes through a fixed point.

184. From any curve, various others can be derived, such as the inverse, the pedal, the polar reciprocal, and the evolute.

The *inverse* of a point with respect to a given circle is defined in § 58; if O is the centre of the circle, k the radius, the inverse of a point P is a point Q on the line OP, such that $OP . OQ = k^2$. If P is (x_1, y_1) and $Q(x_2, y_2)$, while OP and OQ are r_1, r_2, then—

$$x_1 = r_1 \cos\theta,\ y_1 = r_1 \sin\theta,$$
$$x_2 = r_2 \cos\theta,\ y_2 = r_2 \sin\theta,$$
$$\cos^2\theta = \frac{x_1^2}{x_1^2 + y_1^2} = \frac{x_2^2}{x_2^2 + y_2^2},$$

hence $$x_1x_2 = k^2\cos^2\theta,\ y_1y_2 = k^2\sin^2\theta.$$

These give $$x_1 = \frac{k^2x_2}{x_2^2 + y_2^2},\ y_1 = \frac{k^2y_2}{x_2^2 + y_2^2},$$

and also $$x_2 = \frac{k^2x_1}{x_1^2 + y_1^2},\ y_2 = \frac{k^2y_1}{x_1^2 + y_1^2}.$$

From these formulæ, when the curve described by P (the locus of x_1, y_1) is known, the curve described by Q (the locus of x_2, y_2) can be found. These two curves are said to be inverse with respect to the circle; or, frequently, with respect to the point O.

185. The *pedal* of a curve with respect to a point O is the locus of the foot of the perpendicular from O to a

tangent. If $y = mx + n$ is the tangent, so that n is known in terms of m, the pedal with respect to the origin is found by eliminating m from the two equations—

$y = mx + n$, in which n is expressed in terms of m,

$x + my = 0$, the equation of the line through the origin perpendicular to the tangent.

For example, the tangent to a central conic,

$$\frac{x^2}{a} + \frac{y^2}{\beta} = 1,$$

is

$$y = mx + \sqrt{am^2 + \beta};$$

the line through the origin perpendicular to the tangent is—

$$x + my = 0;$$

hence the pedal is $$y = -\frac{x^2}{y} + \sqrt{a\frac{x^2}{y^2} + \beta},$$

that is, $$x^2 + y^2 = \sqrt{ax^2 + \beta y^2},$$

or $$(x^2 + y^2)^2 = ax^2 + \beta y^2.$$

If the line-equation of a curve is $f(\xi, \eta) = 0$, the tangent is $\xi x + \eta y + 1 = 0$; the line through O perpendicular to this tangent is $\frac{x}{\xi} - \frac{y}{\eta} = 0$. The pedal is found by eliminating ξ, η from these three equations. The two last give—

$$\xi = \frac{-x}{x^2 + y^2}, \quad \eta = \frac{-y}{x^2 + y^2},$$

hence the equation of the pedal is—

$$f\left(\frac{-x}{x^2 + y^2}, \frac{-y}{x^2 + y^2}\right) = 0.$$

For example, the line-equation of $\frac{x^2}{a} + \frac{y^2}{\beta} = 1$ is—

$$a\xi^2 + \beta\eta^2 = 1.$$

Hence the pedal with respect to the origin is—

$$\frac{\alpha x^2}{(x^2+y^2)^2} + \frac{\beta y^2}{(x^2+y^2)^2} = 1,$$

that is,

$$(x^2+y^2)^2 = \alpha x^2 + \beta y^2.$$

186. *Polar reciprocal.*—The pole, with respect to a fixed conic, of a tangent to a given curve describes a curve called the polar of the given curve with respect to the fixed conic. If U is the intersection of two tangents p, q, whose poles with respect to the fixed conic are P, Q, then U is the pole of PQ (§ 102). When the lines p, q become indistinguishable, and therefore also the points P, Q, the point U is a point on the given curve, and PQ is a tangent to the derived curve, the polar (Fig. 95). Hence the polar,

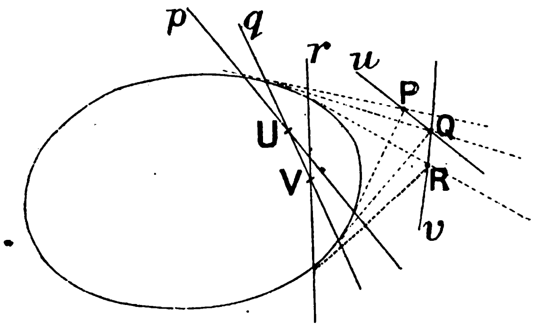

Fig. 95.

obtained as the locus of the pole P of a tangent p to the given curve, is also the envelope of the polar u of a point U on the given curve; and the given curve is the locus of the pole U of a tangent u to the derived curve, and also the envelope of the polar p of a point P on the derived

curve. The relation between the given curve and the derived curve is thus perfectly symmetrical; each is the locus of the pole (with respect to a fixed conic) of a variable tangent to the other, and each is the envelope of the polar of a variable point on the other. The two are said to be reciprocal polars, or polar reciprocals, with respect to the fixed conic.

To find the point-equation of the polar reciprocal of a curve f, with respect to a given conic, for instance

$$\frac{x^2}{\alpha} + \frac{y^2}{\beta} = 1,$$

take a tangent to f, $\xi x + \eta y + 1 = 0$, and find its pole with respect to $\frac{x^2}{\alpha} + \frac{y^2}{\beta} = 1$. If this is (x', y') the line must be $\frac{xx'}{\alpha} + \frac{yy'}{\beta} - 1 = 0$, hence $\xi = -\frac{x'}{\alpha}$, $\eta = -\frac{y'}{\beta}$. If then the relation that connects ξ, η is known, as it certainly is since f is a known curve, the relation that connects x', y' is found.

For example, find the polar reciprocal of the general conic,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

with respect to $\frac{x^2}{\alpha} + \frac{y^2}{\beta} = 1$.

The line-equation of the general conic is—

$$A\xi^2 + 2H\xi\eta + B\eta^2 + 2G\xi + 2F\eta + C = 0;$$

by means of the relations $\xi = -\frac{x'}{\alpha}$, $\eta = -\frac{y'}{\beta}$, this gives for the locus of (x', y')—

$$\frac{A}{\alpha^2}x^2 + 2\frac{H}{\alpha\beta}xy + \frac{B}{\beta^2}y^2 - 2\frac{G}{\alpha}x - 2\frac{F}{\beta}y + C = 0.$$

Hence the polar reciprocal is a conic.

If the line-equation is desired, we obtain it as the envelope of the polar of a point that describes the curve f. We have then—

$$ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0,$$

where $$x_1 = -\alpha\xi,\ y_1 = -\beta\eta.$$

Hence the envelope is—

$$a\alpha^2\xi^2 + 2h\alpha\beta\xi\eta + b\beta^2\eta^2 - 2g\alpha\xi - 2f\beta\eta + c = 0.$$

187. The *evolute* of a curve is the envelope of the normals. If the normals at P, Q meet at U (Fig. 96), and

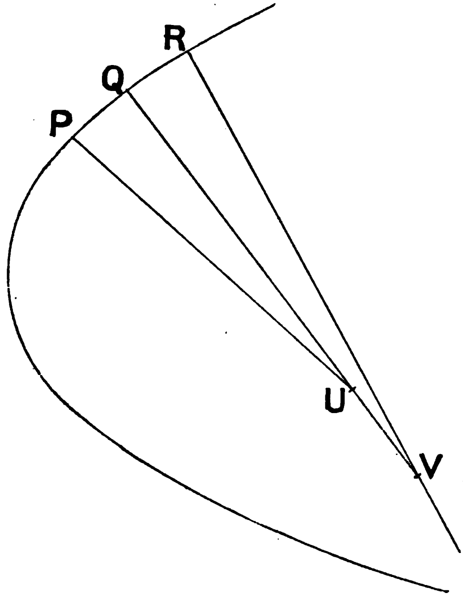

Fig. 96.

the normals at Q, R at V, then when PQR are close together on the given curve, the points U, V are the intersections of indistinguishable tangents to the evolute,

hence they are points on the evolute. That is, the evolute is the locus of the point of intersection of indistinguishable (or consecutive) normals. The point at which the normal at P is met by the consecutive normal is called the centre of curvature at P; hence the evolute of a curve is the locus of the centre of curvature. (See §§ 141, xiii; 144, xvii.)

EXAMPLES.

1. Find the inverse of a conic with respect to a focus.
2. Find the pedal of a conic with respect to a focus.
3. Prove that the polar reciprocal of a conic through the centre of the fixed conic is a parabola.
4. Show that the line (h, k) is the polar of the point (h, k) with respect to the imaginary circle $x^2 + y^2 + 1 = 0$.
5. Prove that the polar reciprocal of a conic with one focus and directrix the same as those of the fixed conic is a conic with the same focus and directrix.
6. Three conics have a common focus and directrix. Prove that two are polar reciprocals with respect to the other if the three eccentricities are in geometric progression.
7. The line OP is produced to Q, so that PQ is of a given length b. Express the coordinates of P in terms of the coordinates of Q. Hence find the locus of Q when P describes the circle $x^2 + y^2 - ax = 0$. Also when P describes a straight line parallel to Oy.
8. Find the pedal of a circle with respect to a given point—(i) on the circle, (ii) external to the circle, (iii) internal to the circle.
9. Find the coordinates of a chord of a central conic in terms of the coordinates of the point of bisection, and *vice-versa*. Hence find the envelope of the chord if the point of bisection describes a straight line, not passing through the centre.
10. Apply the formulæ of Ex. 9 to find the locus of the point of bisection of a chord through a fixed point.

CHAPTER XIII

MISCELLANEOUS EXAMPLES

188. THIS chapter contains applications of the methods already explained to several examples on conics. Many of these are standard problems; the hints as to methods conveyed in the solutions should be attended to.

The Parabola.

Example 1.—If a triangle circumscribes a parabola, the orthocentre is on the directrix, and the circumscribing circle passes through the focus.

(i) If the slopes of the three sides are m_1, m_2, m_3, their equations are—

$$y = m_1x + \frac{p}{m_1},\ y = m_2x + \frac{p}{m_2},\ y = m_3x + \frac{p}{m_3};$$

the intersections of these are $\left[\dfrac{p}{m_2m_3},\ p\left(\dfrac{1}{m_2} + \dfrac{1}{m_3}\right)\right]$, etc. It is however often more convenient to use the slope of the normal to a parabola than the slope of the tangent, that is, to write $n_1 = -\dfrac{1}{m_1}$, $n_2 = -\dfrac{1}{m_2}$, $n_3 = -\dfrac{1}{m_3}$. The three sides are then $y = -\dfrac{x}{n_1} - pn_1$, etc., and the vertices are $(pn_2n_3,\ -p[n_2 + n_3])$, etc.

The line through the first vertex perpendicular to the opposite side is—

$$n_1(x - pn_2n_3) = y + p(n_2 + n_3),$$

that is, $$n_1(x + p) = y + p[(n_1 + n_2 + n_3) + n_1n_2n_3].$$

If we denote the symmetric functions of the three n's by N_1, N_2, N_3, this becomes—

$$n_1(x + p) = y + p(N_1 + N_3).$$

The other two perpendiculars are—

$$n_2(x + p) = y + p(N_1 + N_3),$$
$$n_3(x + p) = y + p(N_1 + N_3).$$

These three lines all pass through the point at which $x + p = 0$, $y + p(N_1 + N_3) = 0$; hence the orthocentre of the triangle is $(-p, -p[N_1 + N_3])$, which lies on the directrix $x + p = 0$.

(ii) The equation of any circle is—

$$x^2 + y^2 + 2gx + 2fy + c = 0;$$

f, g, c are to be chosen so as to make this pass through the vertices of the triangle. It meets the line—

$$y = -\frac{1}{n_1}x - pn_1,$$

that is, $$x = -n_1y - n_1^2p$$

where $(n_1y + n_1^2p)^2 + y^2 - 2g(n_1y + n_1^2p) + 2fy + c = 0$,

or

$$(n_1^2 + 1)y^2 + 2(n_1^3p - n_1g + f)y + n_1^4p^2 - 2n_1^2pg + c = 0;$$

the roots of this are to be $-p(n_1+n_2)$, $-p(n_1+n_3)$, hence

$$p(2n_1+n_2+n_3)(n_1^2+1) = 2n_1^3p - 2n_1g + 2f,$$

$$p^2(n_1^2+n_1n_2+n_1n_3+n_2n_3)(n_1^2+1) = n_1^4p - 2n_1^2pg + c,$$

that is, $p(n_1+N_1)(n_1^2+1) = 2n_1^3p - 2n_1g + 2f,$

$$p^2(n_1^2+N_2)(n_1^2+1) = n_1^4p^2 - 2n_1^2pg + c,$$

or $\quad -n_1^3p + n_1^2N_1p + n_1(p+2g) + N_1p - 2f = 0,$

$$n_1^2(p^2+N_2p^2+2pg) + N_2p^2 - c = 0.$$

Each of these two equations must hold also when n_2 or n_3 is written for n_1. The second shows therefore that—

$$N_2p^2 - c = 0,\ p^2 + N_2p^2 + 2pg = 0.$$

These give $c = N_2p^2$ and $p^2 + 2pg + c = 0$; this last is the condition that the circle passes through the focus $(p, 0)$.

Example 2.—In proving the second part of No. 1, it is not necessary to find the equation of the circumscribing circle, but as this will be needed shortly, we proceed to find it. We have found $c = N_2p^2$, $p^2 + N_2p^2 + 2pg = 0$, from which $2g = -p(1+N_2)$. Also the equation

$$n^3p - n^2N_1p - n(p+2g) - N_1p + 2f = 0$$

holds when for n is written n_1, n_2, or n_3. Now the equation

$$n^3 - n^2N_1 + nN_2 - N_3 = 0$$

has n_1, n_2, n_3 for its roots; hence these two equations are identical. That is—

$$\frac{p}{1} = \frac{N_1p}{N_1} = \frac{p+2g}{-N_2} = \frac{N_1p-2f}{N_3},$$

or $\quad p + 2g = -N_2p,\ N_1p - 2f = N_3p.$

The value for $2g$, $-p(1 + N_2)$, has already been found; the remaining equation gives $2f = p(N_1 - N_3)$, hence the equation of the circumscribing circle is—

$$x^2 + y^2 - p(1 + N_2)x + p(N_1 - N_3)y + p^2N_2 = 0.$$

189. *Example* 3.—Three normals to a parabola pass through a point (h, k); the points on the parabola at which these normals are drawn are determined by a certain rectangular hyperbola, and the circle through the points passes through the vertex of the parabola.

(i) The normal at (x_1, y_1) is—

$$y_1(x - x_1) + 2p(y - y_1) = 0;$$

this passes through (h, k) if

$$y_1(h - x_1) + 2p(k - y_1) = 0.$$

The point (x_1, y_1) must therefore lie on—

$$y(h - x) + 2p(k - y) = 0,$$

that is, on the rectangular hyperbola—

$$xy + (2p - h)y - 2pk = 0.$$

The ordinates of the common points of this hyperbola and the parabola $y^2 = 4px$ are given by—

$$\frac{y^2}{4p}y + (2p - h)y - 2pk = 0,$$

that is, by the cubic equation—

$$y^3 + 4p(2p - h)\,y - 8p^2k = 0;$$

hence there are three points whose normals pass through (h, k), and the sum of the three ordinates is zero.

The equation for the slopes of the three normals is (since $y_1 = -2pn_1$)—

$$pn^3 + (2p - h)n + k = 0.$$

(ii) The circle—

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

meets the parabola where

$$\frac{y^4}{16p^2} + y^2 + \frac{2gy^2}{4p} + 2fy + c = 0,$$

that is, where

$$y^4 + (16p^2 + 8pg)y^2 + 32p^2fy + 16p^2c = 0.$$

Hence the sum of the ordinates of the four points $= 0$. Now the sum of the ordinates of the three points determined by the normals from $(h, k) = 0$, hence the ordinate of the fourth point $= 0$, and the point is consequently the vertex. Hence $c = 0$.

Example 4.—Find the equation of the circle through the three points determined by the normals from hk.

The ordinates of the points are given by—

$$y^3 + (16p^2 + 8pg)y + 32p^2f = 0,$$

and also by $y^3 + 4p(2p - h)y - 8p^2k = 0$.

Hence $\qquad 16p^2 + 8pg = 4p(2p - h),$

and $\qquad 32p^2f = -8p^2k.$

From these $\qquad 2g = -(2p + h),$

and $\qquad 2f = -\dfrac{k}{2};$

hence the equation of the circle through the three points whose normals pass through (h, k) is—

$$x^2 + y^2 - (2p + h)x - \frac{k}{2}y = 0.$$

Moreover, the circle $x^2 + y^2 + 2gx + 2fy = 0$ (that is, any circle through the vertex) meets the parabola in three points, and the normals at these points meet at the point $(-2[p+g], -4f)$.

Three points the normals at which are concurrent are called conormal points; the point at which the normals meet has no recognised name, but for convenience it will here be called the normal point for the triangle. Also the point of intersection of the normals at the extremities of a chord will be called the normal point for the chord.

The chord that joins the points n_1, n_2 has the equation—

$$2x + (n_1 + n_2)y + 2pn_1n_2 = 0.$$

The normals at n_1, n_2 are—

$$n_1(x - 2p) - y = pn_1^3,$$
$$n_2(x - 2p) - y = pn_2^3;$$

these meet at $x = 2p + p(n_1^2 + n_1n_2 + n_2^2)$,

$$y = pn_1n_2(n_1 + n_2).$$

Hence the normal point for the line (ξ, η) is—

$$x = 2p + p\left(\frac{4\eta^2}{\xi^2} - \frac{1}{p\xi}\right), \quad y = \frac{2\eta}{\xi^2};$$

that is, $\left(\dfrac{2p\xi^2 + 4p\eta^2 - \xi}{\xi^2}, \dfrac{2\eta}{\xi^2}\right)$.

If the pole of the line is (x', y'), the equation of the line is—

$$yy' = 2p(x + x'),$$

or $$\frac{1}{x'} \cdot x - \frac{y'}{2px'} \cdot y + 1 = 0;$$

hence $\xi = \frac{1}{x'}$, $\eta = -\frac{y'}{2px'}$, and the coordinates of the normal point become $\left(2p + \frac{y'^2}{p} - x',\ -\frac{x'y'}{p}\right)$.

These expressions are of use in problems relating to normals to a parabola.

Example 5.—If the points P, Q at which the normals are drawn are indistinguishable, the chord PQ is a tangent; the point of intersection of the normals is called the centre of curvature at P. Hence the coordinates of the centre of curvature at x_1, y_1 are—

$$x = 2p + \frac{y_1^2}{p} - x_1,\ y = -\frac{x_1y_1}{p}.$$

Since $x_1 = \frac{y_1^2}{4p}$, these become—

$$x = 2p + \frac{3y_1^2}{4p},\quad y = -\frac{y_1^3}{4p^2}.$$

To find the locus of the centre of curvature, eliminate y_1. The two equations give—

$$4p(x - 2p) = 3y_1^2,$$

$$4p^2y = -y_1^3,$$

hence $$[4p(x - 2p)]^3 = 27(4p^2y)^2,$$

that is, $$4(x - 2p)^3 = 27py^2.$$

190. *Example* 6.—A chord passes through a fixed point on the axis of the parabola; find the locus of the normal point.

A line $\xi x + \eta y + 1 = 0$ passes through the fixed point

$(q, 0)$ if $q\xi + 1 = 0$; hence $\xi = -\dfrac{1}{q}$, while η varies. The coordinates of the normal point are—

$$x = 2p + 4p\frac{\eta^2}{\xi^2} - \frac{1}{\xi} = 2p + 4pq^2\eta^2 + q,$$

$$y = 2\frac{\eta}{\xi^2} = 2q^2\eta.$$

To find the locus, eliminate η; the result is—

$$x = 2p + 4pq^2 \cdot \frac{y^2}{4q^4} + q$$

$$= 2p + \frac{py^2}{q^2} + q,$$

hence $\quad y^2 = \dfrac{q^2}{p}(x - 2p - q).$

The locus is therefore a parabola. In particular, if $q = -2p$, it is the original parabola itself; hence if a chord of a parabola passes through the point $(-2p, 0)$, the normals at its extremities meet on the parabola.

The equation $py^2 = q^2(x - 2p - q)$ shows that for a given normal point (h, k), q satisfies the equation—

$$q^3 + (2p - h)q^2 + pk^2 = 0;$$

the three values of q belong to the three sides of the conormal triangle.

Example 7.—Find the locus of a point if one normal from it bisects the angle between the other two.

The slopes of the three normals from (h, k) are given by—

$$pn^3 + (2p - h)n + k = 0.$$

If the inclinations of the normals are θ_1, θ_2, θ_3, the given condition is—

$$2\theta_3 = \theta_1 + \theta_2,$$

hence $\quad \tan 2\theta_3 = \tan(\theta_1 + \theta_2),$

or $$\frac{2n_3}{1 - n_3^2} = \frac{n_1 + n_2}{1 - n_1 n_2} = \frac{-n_3}{1 - n_1 n_2}, \text{ since } \Sigma n = 0.$$

This gives $\quad 2(1 - n_1 n_2) + 1 - n_3^2 = 0,$

or $\quad 3 - 2n_1 n_2 - n_3^2 = 0,$

from which $\quad 3 - 2n_1 n_2 + n_3(n_1 + n_2) = 0,$

hence $\quad 3 - 3n_1 n_2 + \mathrm{N}_2 = 0,$

that is, $$3 - 3n_1 n_2 + \frac{2p - h}{p} = 0.$$

Hence $$n_1 n_2 = \frac{5p - h}{3p},$$

and therefore $$n_3 = \frac{n_1 n_2 n_3}{n_1 n_2} = \frac{-3k}{5p - h}.$$

Now n_3 satisfies—

$$pn^3 + (2p - h)n + k = 0,$$

hence $$-\frac{27pk^3}{(5p - h)^3} - \frac{3k(2p - h)}{5p - h} + k = 0,$$

that is, $$27pk^2 = (5p - h)^2[5p - h - 3(2p - h)]$$
$$= (5p - h)^2[2h - p].$$

Hence the locus of the point (h, k) is—

$$(2x - p)(x - 5p)^2 = 27py^2.$$

191. *Example* 8.—Find the orthocentre, the centroid, and the circumcentre of a conormal triangle. Also of the triangle formed by the tangents at the conormal points.

(i) The conormal triangle.

The chord n_2n_3 is

$$2x + (n_2 + n_3)y + 2pn_2n_3 = 0;$$

the line through n_1 perpendicular to this chord is—

$$(n_2 + n_3)(x - pn_1^2) - 2(y + 2pn_1) = 0,$$

that is, since $\Sigma n = 0$,

$$n_1(x - pn_1^2) + 2(y + 2pn_1) = 0,$$

with similar equations for the other two lines by which the orthocentre is determined. Hence if (x, y) is the orthocentre, the equation—

$$pn^3 - (x + 4p)n - 2y = 0$$

is satisfied by n_1, n_2, n_3, the roots of

$$pn^3 + (2p - h)n + k = 0.$$

Hence $\qquad - (x + 4p) = 2p - h,$

and $\qquad - 2y = k;$

the orthocentre is therefore $\left(h - 6p, -\frac{k}{2}\right)$.

The coordinates of the centroid are $x = \frac{1}{3}\Sigma x_1$, $y = \frac{1}{3}\Sigma y_1$.

Now $x_1 = pn_1^2$, $y_1 = -2pn_1$; hence $x = \frac{p}{3}\Sigma n_1^2$, $y = -\frac{2p}{3}\Sigma n_1$.

Since $\qquad \Sigma n_1 = 0$ and $\Sigma n_1 n_2 = \dfrac{2p - h}{p}$,

$$\Sigma n_1^2 = (\Sigma n_1)^2 - 2\Sigma n_1 n_2 = -2\frac{2p - h}{p},$$

hence the centroid is $\left(-\frac{2(2p - h)}{3}, 0\right)$.

The equation of the circle through three conormal points has been found to be—

$$x^2 + y^2 - (2p + h)x - \frac{k}{2}y = 0,$$

hence the circumcentre is $\left(\frac{2p + h}{2}, \frac{k}{4}\right)$.

(ii) The circumscribed triangle.

The orthocentre of any circumscribed triangle has been shown to be $(-p, -p(N_1 + N_3))$, hence the orthocentre is $(-p, k)$.

The vertices are $(pn_2n_3, -p(n_2 + n_3))$, etc., hence the centroid is $\left(\frac{1}{3}p\Sigma n_2n_3, -\frac{1}{3}p\Sigma(n_2 + n_3)\right)$, that is, $\left(\frac{2p - h}{3}, 0\right)$.

The equation of the circle through the vertices of the triangle formed by three tangents has been found to be—

$$x^2 + y^2 - p(1 + N_2)x + p(N_1 - N_3)y + p^2N_2 = 0.$$

For conormal points $N_1 = 0$, $N_2 = \frac{2p - h}{p}$, $N_3 = -\frac{k}{p}$, and the circle is therefore—

$$x^2 + y^2 - (3p - h)x + ky + p(2p - h) = 0.$$

Hence the circumcentre is $\left(\frac{3p - h}{2}, \frac{-k}{2}\right)$.

192. *Example* 9.—If the vertices of an inscribed triangle lie on $$xy + \lambda x + \mu y + \nu = 0,$$
the vertices of the corresponding circumscribed triangle lie on $$2xy + \lambda x - \nu = 0,$$
and also on $$y^2 + \frac{\lambda}{2}y = px - p\mu.$$

The tangents at (x_2, y_2), (x_3, y_3) meet at (x', y'), where $x' = \frac{y_2 y_3}{4p}$, $y' = \frac{y_2 + y_3}{2}$. The hyperbola—

$$xy + \lambda x + \mu y + \nu = 0$$

meets the parabola where

$$y^3 + \lambda y^2 + 4p\mu y + 4p\nu = 0.$$

Hence $\Sigma y = -\lambda$, $y_1 y_2 y_3 = -4p\nu$, values which make $x' = -\frac{\nu}{y_1}$, $y' = -\frac{y_1 + \lambda}{2}$. Eliminating y_1 from these we find—

$$x'(2y' + \lambda) = \nu;$$

that is, any vertex of the circumscribed triangle lies on the hyperbola—

$$2xy + \lambda x - \nu = 0.$$

Moreover $$\Sigma y_1 y_2 = 4p\mu,$$

that is, $$y_2 y_3 + y_1(y_2 + y_3) = 4p\mu.$$

Hence $$4px' - (2y' + \lambda)2y' = 4p\mu.$$

Any vertex lies therefore on $4y^2 + 2\lambda y = 4px - 4p\mu$,

that is, on $$y^2 + \frac{\lambda}{2}y = px - p\mu.$$

Similarly it can be shown that the vertices of the circumscribed triangle lie on $x^2 - \mu x = \frac{\nu}{2p}y$.

If the triangle is conormal, $\lambda = 0$; the inscribed (conormal) triangle is cut out by—

$$xy + \mu y + \nu = 0,$$

the circumscribed triangle is determined by—

$$2xy - \nu = 0,$$

and

$$y^2 = p(x - \mu),$$

or by

$$x^2 - \mu x = \frac{\nu}{2p} y.$$

If (h, k) is the normal point, $2p - h = \mu, \ - 2pk = \nu$.

Examples.

1. Normals PQ, PR are drawn from a point P on a parabola. Find the locus of the circumcentre and orthocentre of the triangle PQR as P moves along the parabola.

2. Find the locus of the circumcentre of the triangle formed by the tangents at P, Q, R (Ex. 1).

3. Find the locus of the normal point for a conormal triangle PQR if the circle PQR is of constant radius. Prove that for different values of the radius the loci are a series of similar similarly situated concentric ellipses $4(x + 2a)^2 + y^2 =$ constant.

4. Find the locus of the normal point for a conormal triangle PQR if the circumcircle of the circumscribed triangle is of constant radius.

5. If the locus of the normal point for a chord is a straight line perpendicular to the axis, the locus of the pole is a parabola.

6. If the locus of the normal point for a chord is a diameter, the locus of the pole is a rectangular hyperbola.

7. Prove that of all the triangles circumscribed to a parabola and inscribed to a given circle through the focus there is precisely one whose sides determine a set of conormal points. If the centre of the circle is (a, b) and the normal point (h, k), then—

$$\begin{aligned} h - 3p &= -2a, \\ k &= -2b. \end{aligned}$$

8. Find the locus of the circumcentre and orthocentre of a conormal triangle if the normal pole describes a straight line.

9. The normal pole of PQR describes the line $x = 5p$; prove that the orthocentre of PQR is the circumcentre of the triangle formed by the tangents at P, Q, R, and lies on the directrix.

10. If the ordinates of the vertices of an inscribed triangle satisfy the equations $y^3 - \lambda y^2 + \mu y - \nu = 0$, the abscissæ satisfy—

$$64p^3x^3 - 16p^2(\lambda^2 - 2\mu)x^2 + 4p(\mu^2 - 2\lambda\nu)x - \nu^2 = 0,$$

and the coordinates of the vertices of the circumscribed triangle satisfy—

$$8y^3 - 8\lambda y^2 + 2(\lambda^2 + \mu)y - (\lambda\mu - \nu) = 0,$$
$$64p^3x^3 - 16p^2\mu x^2 + 4p\lambda\nu x - \nu^2 = 0.$$

11. Prove that the distance from the focus to the normal point for a conormal triangle is equal to a diameter of the circle through the poles of the sides of the triangle.

12. Prove that a rectangular hyperbola which has the axis of a parabola for an asymptote meets the parabola at three finite points whose normals (to parabola) are concurrent in a point on the hyperbola.

13. Prove that a rectangular hyperbola which has the tangent at the vertex of a parabola as one asymptote meets the parabola in three points P′, Q′, R′ such that the other extremities of the focal chords through P′, Q′, R′ are conormal.

14. If the centroid of an inscribed triangle PQR moves along the axis the focal chords through P, Q, R meet the parabola again at the vertices of a triangle whose centroid describes the parabola $y^2 = \frac{4}{3}px$.

15. Find a point through which pass three equally inclined normals.

16. Show that a conormal triangle cannot be equilateral.

17. Find the locus of the circumcentre of a right-angled conormal triangle.

18. Find the locus of the normal point for a right-angled conormal triangle.

19. Prove that the sides of a conormal triangle meet the axis at points whose abscissæ q, q', q'' are connected by the relation $\Sigma\frac{1}{q} = 0$. Find the coördinates of the normal point in terms of q, q', q''.

20. Find a conormal triangle two of whose sides shall pass through given points on the axis.

21. The axis of a parabola meets the sides of a conormal triangle at three points, one of which lies half-way between the other two. Find the locus of the normal point.

22. A parabola of given size moves without rotation, so that the

focus describes a circle. Find the locus of the normal point for the circumscribed conormal triangle which is inscribed to the circle.

23. A parabola revolves about its focus. Find the locus of the normal point which is determined by a fixed circle through the focus. (Cf. Ex. 7.)

24. The circumcentre of a circumscribed triangle (conormal) describes a circle whose centre is at the focus. Find the locus of the normal point.

25. Find the locus of the points of contact of curves of the system $xy = \lambda$ with curves of the system $y^2 - px = \mu$.

26. A parabola of given size slides through the origin without rotation. Show that the locus of the vertex is an equal parabola with its concavity in the opposite direction. Find also the locus of the focus.

27. Find the equation of the circle of curvature at a point (x_1, y_1) on the curve. Show that of the four intersections with the parabola three lie at (x_1, y_1), one at $(9x_1, -3y_1)$.

Note.—The circle of curvature at (x_1, y_1) has its centre at the centre of curvature, and passes through (x_1, y_1).

Normals to a Central Conic.

193. *Example* 10.—Four normals to a central conic pass through a point (h, k); the four conormal points are determined by a certain rectangular hyperbola.

The equation of the normal to $\frac{x^2}{\alpha} + \frac{y^2}{\beta} = 1$ at (x_1, y_1) is—

$$\frac{\alpha x}{x_1} - \frac{\beta y}{y_1} = \alpha - \beta.$$

Hence the normal at (x, y) passes through (h, k) if

$$\frac{\alpha h}{x} - \frac{\beta k}{y} = \alpha - \beta,$$

that is, if $(\alpha - \beta)xy + \beta kx - \alpha hy = 0.$

The point (x, y) lies on this rectangular hyperbola, which

passes through the centre of the given conic, and also through the point (h, k). Since the hyperbola meets the given conic in four points, there are four normals through (h, k).

The pencil of conics through the four conormal points is—

$$\frac{x^2}{\alpha} + \frac{y^2}{\beta} - 1 + \lambda[(\alpha - \beta)xy + \beta kx - \alpha hy] = 0.$$

If $(\xi x + \eta y + 1)(\xi' x + \eta' y + 1) = 0$ is one of the line pairs contained in this pencil, comparison of coefficients shows that—

$$\xi\xi' : \frac{1}{\alpha} = \eta\eta' : \frac{1}{\beta} = 1 : -1,$$

hence $\xi\xi' = -\frac{1}{\alpha}$, $\eta\eta' = -\frac{1}{\beta}$. If one chord is taken arbitrarily (that is, if two of the four points are chosen), the other chord is determined by these equations.

Two chords which pass through four conormal points will be called conormal chords.

Example 11.—Find the normal point for a chord in terms of the coordinates of the extremities. Also in terms of the coordinates of the chord itself. Hence find the coordinates of the centre of curvature, and the locus of the centre of curvature.

(i) The chord (x_1, y_1), (x_2, y_2) can be written in either of the forms—

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2},$$

that is, $\quad (y_1 - y_2)x - (x_1 - x_2)y + x_1y_2 - x_2y_1 = 0,$

and $\quad \frac{x_1 + x_2}{\alpha}x + \frac{y_1 + y_2}{\beta}y - \left(1 + \frac{x_1x_2}{\alpha} + \frac{y_1y_2}{\beta}\right) = 0.$

The normals at (x_1, y_1), (x_2, y_2) meet at—

$$x = \frac{\alpha - \beta}{\alpha} . \frac{x_1x_2(y_1 - y_2)}{x_2y_1 - x_1y_2}, \quad y = \frac{\alpha - \beta}{\beta} . \frac{y_1y_2(x_1 - x_2)}{x_2y_1 - x_1y_2}.$$

Since $\dfrac{y_1 - y_2}{x_2y_1 - x_1y_2} = -\xi$, and $\dfrac{x_1 - x_2}{x_2y_1 - x_1y_2} = \eta$, these values for the coordinates of the normal point can be written in the compact form—

$$\left(-\frac{\alpha - \beta}{\alpha}x_1x_2\xi, \ \frac{\alpha - \beta}{\beta}y_1y_2\eta\right),$$

or again, since

$$\xi = \frac{-\dfrac{x_1 + x_2}{\alpha}}{1 + \dfrac{x_1x_2}{\alpha} + \dfrac{y_1y_2}{\beta}}, \quad \eta = \frac{-\dfrac{y_1 + y_2}{\beta}}{1 + \dfrac{x_1x_2}{\alpha} + \dfrac{y_1y_2}{\beta}},$$

in the form—

$$\left(\frac{\alpha - \beta}{\alpha^2} . \frac{(x_1 + x_2)x_1x_2}{1 + \dfrac{x_1x_2}{\alpha} + \dfrac{y_1y_2}{\beta}}, \ -\frac{\alpha - \beta}{\beta^2} . \frac{(y_1 + y_2)y_1y_2}{1 + \dfrac{x_1x_2}{\alpha} + \dfrac{y_1y_2}{\beta}}\right).$$

To express these in terms of ξ, η it is necessary to express x_1x_2, y_1y_2. The line $\xi x + \eta y + 1 = 0$ meets $\dfrac{x^2}{\alpha} + \dfrac{y^2}{\beta} = 1$ where

$$x^2\frac{\eta^2}{\alpha} + \frac{(\xi x + 1)^2}{\beta} - \eta^2 = 0,$$

or $$(\alpha\xi^2 + \beta\eta^2)x^2 + 2\alpha\xi x + \alpha(1 - \beta\eta^2) = 0.$$

Hence $$x_1x_2 = \frac{\alpha(1 - \beta\eta^2)}{\alpha\xi^2 + \beta\eta^2},$$

and similarly $$y_1y_2 = \frac{\beta(1 - \alpha\xi^2}{\alpha\xi^2 + \beta\eta^2}.$$

The normal point for the chord $\xi x + \eta y + 1 = 0$ is therefore $\left(-(\alpha - \beta)\xi\frac{1 - \beta\eta^2}{\alpha\xi^2 + \beta\eta^2}, (\alpha - \beta)\eta\frac{1 - \alpha\xi^2}{\alpha\xi^2 + \beta\eta^2} \right)$.

(ii) To find the centre of curvature, make the points (x_1, y_1), (x_2, y_2) indistinguishable, or make $\xi x + \eta y + 1 = 0$ a tangent. This requires $\alpha\xi^2 + \beta\eta^2 = 1$; hence for the centre of curvature—

$$x = -(\alpha - \beta)\xi(1 - \beta\eta^2) = -(\alpha - \beta)\alpha\xi^3,$$
$$y = (\alpha - \beta)\eta(1 - \alpha\xi^2) = (\alpha - \beta)\beta\eta^3,$$

where $\xi = -\frac{x_1}{\alpha}, \eta = -\frac{y_1}{\beta}$.

Hence $x = \frac{\alpha - \beta}{\alpha^2}x_1^3$, $y = -\frac{\alpha - \beta}{\beta^2}y_1^3$. From either set of values for x, y the locus is obtained, by means of the relation $\alpha\xi^2 + \beta\eta^2 = 1$, or of $\frac{x_1^2}{\alpha} + \frac{y_1^2}{\beta} = 1$; it is—

$$\alpha\left[\frac{x}{\alpha(\alpha - \beta)}\right]^{\frac{2}{3}} + \beta\left[\frac{y}{\beta(\alpha - \beta)}\right]^{\frac{2}{3}} = 1.$$

Example 12.—Find the envelope of the chord which joins the feet of the remaining two normals from the centre of curvature.

This chord is conormal with a tangent. The conditions that $\xi x + \eta y + 1 = 0$ be conormal with the tangent at (x_1, y_1), $\frac{xx_1}{\alpha} + \frac{yy_1}{\beta} = 1$, are—

$$-\xi\frac{x_1}{\alpha} = \frac{-1}{\alpha}, \; -\eta\frac{y_1}{\beta} = -\frac{1}{\beta},$$

that is, $$\xi x_1 = 1, \; \eta y_1 = 1.$$

Hence as (x_1, y_1) describes $\frac{x^2}{\alpha} + \frac{y^2}{\beta} = 1$, the chord (ξ, η) envelopes—

$$\frac{1}{\alpha\xi^2} + \frac{1}{\beta\eta^2} = 1,$$

that is, $$\alpha\beta\xi^2\eta^2 = \alpha\xi^2 + \beta\eta^2.$$

The equation of the chord conormal with the tangent at (x_1, y_1) has thus been shown to be

$$\frac{x}{x_1} + \frac{y}{y_1} + 1 = 0.$$

194. *Example* 13.—Form the equation that gives the slopes of the four normals through (h, k).

The normal at (x_1, y_1) is—

$$\frac{\alpha x}{x_1} - \frac{\beta y}{y_1} = \alpha - \beta,$$

or $$y = \frac{\alpha y_1}{\beta x_1}x - \frac{y_1}{\beta}(\alpha - \beta).$$

The slope is $n = \frac{\alpha y_1}{\beta x_1}$; hence since $\frac{x_1^2}{\alpha} + \frac{y_1^2}{\beta} = 1$,

we have $$\frac{\alpha y_1^2}{\beta^2 n^2} + \frac{y_1^2}{\beta} = 1,$$

from which $y_1^2(\alpha + \beta n^2) = \beta^2 n^2$,

hence $$y_1 = \pm \frac{\beta n}{\sqrt{\alpha + \beta n^2}};$$

and the equation of a normal of slope n is—

$$y = nx \pm \frac{n}{\sqrt{\alpha + \beta n^2}}(\alpha - \beta).$$

The slopes of the normals through (h, k) are therefore given by—

$$k = nh \pm n\frac{\alpha - \beta}{\sqrt{\alpha + \beta n^2}},$$

from which

$$(nh - k)^2 = \frac{n^2(\alpha - \beta)^2}{\alpha + \beta n^2},$$

or $(\beta n^2 + \alpha)(nh - k)^2 - n^2(\alpha - \beta)^2 = 0,$

that is,

$$\beta h^2 n^4 - 2\beta hkn^3 + [\alpha h^2 + \beta k^2 - (\alpha - \beta)^2]n^2 - 2\alpha hkn + \alpha k^2 = 0.$$

Example 14.—Find the slopes of a pair of conormal chords which intersect at (x', y').

The chords are—

$$- mx + y + mx' - y' = 0,$$
$$- m'x + y + m'x' - y' = 0,$$

with the conditions $\xi\xi' = -\frac{1}{\alpha}$, $\eta\eta' = -\frac{1}{\beta}$,

that is,

$$- \alpha mm' = (mx' - y')(m'x' - y')$$
$$- \beta = (mx' - y')(m'x' - y').$$

Hence $mm' = \frac{\beta}{\alpha}$, from which $m' = \frac{\beta}{\alpha m}$. When this value for m' is substituted in either of the conditions found, this becomes—

$$(mx' - y')\left(\frac{\beta x'}{\alpha m} - y'\right) = - \beta,$$

or

$$(mx' - y')(\alpha my' - \beta x') - \alpha\beta m = 0,$$

that is, $\alpha x'y'm^2 - (\beta x'^2 + \alpha y'^2 + \alpha\beta)m + \beta x'y' = 0.$

The roots of this quadratic give the slopes of a pair of conormal chords through (x', y'); and since the equation is only a quadratic, there is only the one pair of chords.

The normal point for the pair of chords can be found by the formulæ proved in Example 11 above, or, more neatly, by the formulæ stated in Example 4, after § 195.

195. *Example* 15.—Find the condition that the three points (x_1, y_1), (x_2, y_2), (x_3, y_3) be conormal.

The three normals are—

$$\frac{\alpha x}{x_1} - \frac{\beta y}{y_1} - (\alpha - \beta) = 0,$$
$$\frac{\alpha x}{x_2} - \frac{\beta y}{y_2} - (\alpha - \beta) = 0,$$
$$\frac{\alpha x}{x_3} - \frac{\beta y}{y_3} - (\alpha - \beta) = 0.$$

The condition that these three lines be concurrent, obtained by the elimination of x, y from the equations, is—

$$\begin{vmatrix} \frac{1}{x_1}, & \frac{1}{y_1}, & 1 \\ \frac{1}{x_2}, & \frac{1}{y_2}, & 1 \\ \frac{1}{x_3}, & \frac{1}{y_3}, & 1 \end{vmatrix} = 0, \text{ or } \begin{vmatrix} x_1, & y_1, & x_1y_1 \\ x_2, & y_2, & x_2y_2 \\ x_3, & y_3, & x_3y_3 \end{vmatrix} = 0.$$

If two of the points are given, this condition yields an equation for either of the two remaining conormal points, namely—

$$\begin{vmatrix} x_1, & y_1, & x_1y_1 \\ x_2, & y_2, & x_2y_2 \\ x, & y, & xy \end{vmatrix} = 0,$$

that is, $xy(x_1y_2 - x_2y_1) - xy_1y_2(x_1 - x_2) + yx_1x_2(y_1 - y_2) = 0$, a rectangular hyperbola through the points (x_1, y_1), (x_2, y_2). Compare Example 10, § 193.

EXAMPLES.

1. Prove that real conormal chords can be parallel only if the conic is an ellipse, and that they are then parallel to one of the equiconjugate diameters.

2. Prove that conormal chords of a rectangular hyperbola are perpendicular.

3. Prove that conormal points on a rectangular hyperbola are the vertices and orthocentre of a triangle.

4. Prove that the normal point for the conormal chords (ξ, η), (ξ', η') is $\left(-\frac{\alpha-\beta}{\alpha}\cdot\frac{\eta+\eta'}{\xi\eta'+\xi'\eta}, \frac{\alpha-\beta}{\beta}\cdot\frac{\xi+\xi'}{\xi\eta'+\xi'\eta}\right)$.

5. Prove that if the normals at P, Q are perpendicular, PQ is a tangent to $\frac{\alpha+\beta}{\alpha^2}x^2+\frac{\alpha+\beta}{\beta^2}y^2=1$.

6. Prove that the locus of the point of intersection of a pair of conormal chords of assigned direction is a hyperbola, whose asymptotes are parallel to the chords.

7. If the intersection of a pair of conormal chords describes the conic $\frac{x^2}{\alpha}-\frac{y^2}{\beta}=1$, the normal point describes the axis of x; and if the intersection describes the conjugate conic $-\frac{x^2}{\alpha}+\frac{y^2}{\beta}=1$, the normal point describes the axis of y.

8. Find the locus of the point of intersection of normals at the extremities of conjugate diameters.

9. If points P, Q, R, T on a hyperbola are conormal, corresponding points on the conjugate hyperbola are conormal, and the line that joins the two normal points is perpendicular to an asymptote.

10. If points P, Q, R, T on an ellipse are conormal, then the corresponding extremities of the conjugate diameters are conormal, and the line that joins the two normal points is perpendicular to one of the equiconjugate diameters.

11. Prove that the feet of the four normals from any point to each of two conjugate hyperbolas are determined by the same rectangular hyperbola.

12. Normals are drawn from a point K to two conjugate hyperbolas. Prove that if the feet of one pair are corresponding points on the two hyperbolas, this holds for all the four pairs; and find the locus of K.

13. If normals at points on an ellipse whose eccentric angles are $\alpha + \beta$, $\alpha - \beta$ are perpendicular, their intersection lies on $\dfrac{x}{a\cos\alpha} = \dfrac{y}{b\sin\alpha}$.

14. Prove that the line $\left(\dfrac{1}{h}, \dfrac{1}{k}\right)$ is conormal with the polar of the point (h, k) for all conics that have the axes of coordinates as axes of symmetry.

15. Prove that the axis of a parabola through four conormal points on an ellipse is parallel to one of the equiconjugate diameters.

16. One pair of chords through four conormal points of an ellipse is parallel if the normal point lies on the diameter perpendicular to either equiconjugate diameter.

17. The product of the distances from the centre of an ellipse to a pair of parallel conormal chords is constant.

18. Find the equation of the centre-locus for the system of conics determined by four conormal points on an ellipse. Prove that the normal to the centre-locus at the centre of the ellipse passes through the point from which the four normals are drawn.

Parametric Treatment of Central Conics.

196. Many properties of the central conics are most simply proved by means of a parametric expression for the coordinates of a point. Any point on an ellipse can be expressed as $\left(a\dfrac{1-t^2}{1+t^2}, b\dfrac{2t}{1+t^2}\right)$, for these expressions satisfy the equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, and moreover they assume all values from $+a$ to $-a$, and from $+b$ to $-b$, as t varies. At the vertices A, A′, $t = 0$ and ∞; at B, B′, $t = 1$ and -1. The parameters of the two extremities of a diameter are connected by the relation $tt' = -1$; for the extremities of conjugate diameters $t' = \dfrac{1+t}{1-t}$, where t'

is in the quadrant immediately succeeding (in the positive sense) the quadrant in which t lies.

The chord t, t' is $\frac{x}{a}(1 - tt') + \frac{y}{b}(t + t') = 1 + tt'$; the pole of this chord is $\left(a\frac{1 - tt'}{1 + tt'}, b\frac{t + t'}{1 + tt'}\right)$. The tangent at t is $\frac{x}{a}(1 - t^2) + \frac{y}{b}2t = 1 + t^2$, and the normal is—

$$\frac{ax}{1 - t^2} - \frac{by}{2t} = \frac{a^2 - b^2}{1 + t^2}.$$

Hence the parameters of points the normals at which pass through (h, k) satisfy—

$$\frac{ah}{1 - t^2} - \frac{bk}{2t} = \frac{a^2 - b^2}{1 + t^2},$$

that is—

$$bkt^4 - (2b^2 - 2ah - 2a^2)t^3 + (2b^2 + 2ah - 2a^2)t - bk = 0.$$

Notice that $\Sigma t_1 t_2 = 0$, $t_1 t_2 t_3 t_4 = -1$.

If (h, k) is itself a point on the curve, with the parameter τ, this equation is divisible by $t - \tau$; the resulting cubic equation for the parameters of three points, the normals at which pass through τ, is—

$$t^3 + \frac{2a^2 - b^2}{b^2\tau}t^2 + \frac{2a^2 - b^2}{b^2}t + \frac{1}{\tau} = 0.$$

For the hyperbola the coordinates are expressed as $\left(a\frac{1 + t^2}{1 - t^2}, b\frac{2t}{1 - t^2}\right)$; the results for the ellipse can be transferred to the hyperbola by writing $-it$, ib for t and b.

The following examples deal in general with three

points on the ellipse, t_1, t_2, t_3; the cubic equation satisfied by the three parameters is written $t^3 - \sigma t^2 + \rho t - \pi = 0$, so that $\sigma = t_1 + t_2 + t_3$, $\rho = t_2t_3 + t_3t_1 + t_1t_2$, $\pi = t_1t_2t_3$.

Example 16.—Find the orthocentre and the circumcentre of the triangle formed by the tangents at t_1, t_2, t_3.

A side of the triangle is $\dfrac{x}{a}(1 - t_1^2) + \dfrac{y}{b}2t_1 = 1 + t_1^2$, and the opposite vertex is $\left(a\dfrac{1 - t_2t_3}{1 + t_2t_3}, b\dfrac{t_2 + t_3}{1 + t_2t_3}\right)$. The line through the vertex perpendicular to the side is—

$$2at_1\left[x - a\frac{1 - t_2t_3}{1 + t_2t_3}\right] + b(t_1^2 - 1)\left[y - b\frac{t_2 + t_3}{1 + t_2t_3}\right] = 0,$$

that is—

$$2at_1[x - a + t_2t_3(x + a)] + b(t_1^2 - 1)[y + yt_2t_3 - b(t_2 + t_3)] = 0.$$

Express this in terms of t_1 and the symmetric functions σ, ρ, π. It becomes—

$$2a(x - a)t_1 + 2a\pi(x + a) + byt_1^2 - by + by\pi t_1 - byt_2t_3 - b^2t_1^2(\sigma - t_1) + b^2(\sigma - t_1) = 0,$$

or, since

$$t_2t_3 = \rho - t_1(t_2 + t_3) = \rho - t_1(\sigma - t_1) = t_1^2 - \sigma t_1 + \rho,$$

$$b^2t_1^3 - b^2\sigma t_1^2 + [2a(x - a) + by\pi + by\sigma - b^2]t_1 + 2a(x + a)\pi - by - by\rho + b^2\sigma = 0.$$

But $t_1^3 = \sigma t_1^2 - \rho t_1 + \pi$, hence the equation reduces to—

$$[2a(x - a) + by(\pi + \sigma) - b^2(1 + \rho)]t_1 + 2a(x + a)\pi - by(1 + \rho) + b^2(\pi + \sigma) = 0.$$

For the second and third lines through the vertices perpendicular to the opposite sides we obtain this same

equation with t_2, t_3 written for t_1. The three lines are therefore of the form $ut_1 + v = 0$, $ut_2 + v = 0$, $ut_3 + v = 0$, which meet at the point $u = 0$, $v = 0$. That is, the orthocentre is given by the equations—

$$2a(x - a) + by(\pi + \sigma) - b^2(1 + \rho) = 0,$$
$$2a(x + a)\pi - by(1 + \rho) + b^2(\pi + \sigma) = 0.$$

If the circumcentre is (g, f), the equation of the circumcircle is $x^2 + y^2 - 2gx - 2fy + c = 0$. This is to pass through the first vertex $\left(a\dfrac{1 - t_2t_3}{1 + t_2t_3}, b\dfrac{t_2 + t_3}{1 + t_2t_3}\right)$, which can be written as $\left(a\dfrac{t_1 - \pi}{t_1 + \pi}, - b\dfrac{t_1^2 - \sigma t_1}{t_1 + \pi}\right)$. Hence—

$$a^2(t_1 - \pi)^2 + b^2(t_1^2 - \sigma t_1)^2 - 2ga(t_1^2 - \pi^2)$$
$$+ 2fb(t_1^2 - \sigma t_1)(t_1 + \pi) + c(t_1 + \pi)^2 = 0.$$

In this, replace the term t_1^4 by $t_1(\sigma t_1^2 - \rho t_1 + \pi)$, and then the term t_1^3 by $\sigma t_1^2 - \rho t_1 + \pi$; thus the equation is reduced to—

$$(- 2ga + 2fb\pi + c + a^2 - b^2\rho)t_1^2$$
$$+ (- 2fb\pi\sigma - 2fb\rho + 2c\pi - 2a^2\pi + b^2\pi + b^2\sigma\rho)t_1$$
$$+ (2ga\pi + 2fb + c\pi + a^2\pi - b^2\sigma)\pi = 0.$$

Similarly the conditions that the circle passes through the second and third vertex are expressed by this same equation with t_2, t_3 written for t_1. That is, we have the three equations—

$$ut_1^2 + vt_1 + w = 0,$$
$$ut_2^2 + vt_2 + w = 0,$$
$$ut_3^2 + vt_3 + w = 0.$$

From these, since t_1, t_2, t_3 are all different, we obtain $u = 0$, $v = 0$, $w = 0$; hence—

$$\begin{aligned}
-2ga + 2fb\pi + c + a^2 - b^2\rho &= 0,\\
-2fb(\pi\sigma + \rho) + 2c\pi - 2a^2\pi + b^2(\pi + \sigma\rho) &= 0.\\
2ga\pi + 2fb + c\pi + a^2\pi - b^2\sigma &= 0.
\end{aligned}$$

These three equations determine the three coefficients f, g, c; for the circumcentre c is not needed, hence by eliminating c we obtain for the circumcentre $x(=g)$, $y(=f)$ the two equations—

$$4a\pi x + 2b(1 - \pi^2)y + b^2(\rho\pi - \sigma) = 0,$$
$$2b(1 + \pi^2 + \rho + \pi\sigma)y + 4a^2\pi - b^2(\sigma + \pi)(1 + \rho) = 0.$$

Note.—If an expression in t of degree less than 3, $ut^2 + vt + w$, (or $vt + w$), is reduced to zero by three values of t, it must vanish identically; for otherwise the quadratic equation $ut^2 + vt + w = 0$ (or the linear equation $vt + w = 0$) would have three roots, which is impossible. Hence $u = 0$, $v = 0$, $w = 0$.

Example 17.—Find the orthocentre and the circumcentre of the inscribed triangle whose vertices are t_1, t_2, t_3.

One side is $\frac{x}{a}(1 - t_2t_3) + \frac{y}{b}(t_2 + t_3) = 1 + t_2t_3$, and the opposite vertex is $-a\frac{t_1^2 - 1}{t_1^2 + 1}$, $2b\frac{t_1}{t_1^2 + 1}$. The line through this vertex perpendicular to the side is—

$$-a(t_2 + t_3)\left(x + a\frac{t_1^2 - 1}{t_1^2 + 1}\right) + b(1 - t_2t_3)\left(y - b\frac{2t_1}{t_1^2 + 1}\right) = 0.$$

Write this in terms of t_1 and the symmetric functions σ, ρ, π. Since however corresponding equations will hold

with t_2, t_3 written for t_1, write t instead of t_1. The equation is—

$$-a(t_2+t_3)[(x+a)t^2+(x-a)] \\ +b(1-t_2t_3)(yt^2-2bt+y)=0,$$

where $t_2+t_3=\sigma-t$, $t_2t_3=t^2-\sigma t+\rho$. Hence—

$$-a(x+a)(\rho t-\pi)+a(x-a)(t-\sigma)+b(yt^2-2bt+y) \\ -by\pi t+2b^2\pi-by(t^2-\sigma t+\rho)=0,$$

that is—

$$[a(x-a)-a(x+a)\rho-by\pi+by\sigma-2b^2]t \\ -a(x-a)\sigma+a(x+a)\pi+by-by\rho+2b^2\pi=0.$$

This equation of the first degree in t is satisfied by all three values of t; hence—

$$a(x-a)-a(x+a)\rho-by\pi+by\sigma-2b^2=0,$$
$$-a(x-a)\sigma+a(x+a)\pi+by-by\rho+2b^2\pi=0;$$

that is, the orthocentre of the inscribed triangle is given by—

$$ax(1-\rho)+by(\sigma-\pi)-a^2(1+\rho)-2b^2=0,$$
$$-ax(\sigma-\pi)+by(1-\rho)+a^2(\sigma+\pi)+2b^2\pi=0.$$

If the circumcentre is (g, f), the equation of the circumcircle is $x^2+y^2-2gx-2fy+c=0$. This is to be satisfied by $x=a\dfrac{1-t^2}{1+t^2}$, $y=b\dfrac{2t}{1+t^2}$, when $t=t_1, t_2, t_3$. Hence

$$a^2(t^2-1)^2+4b^2t^2+2ag(t^4-1)-4bft(t^2+1)+c(t^2+1)^2=0.$$

Expanding this, and replacing t^4 and t^3 by lower powers, as before, we find an equation of the form $ut^2+vt+w=0$. Since this holds for three values of t, we must have

$u = 0$, $v = 0$, $w = 0$; hence the values of f, g, c must satisfy these equations, that is—

$$2ag(\sigma^2 - \rho) - 4bf\sigma + c(\sigma^2 - \rho + 2) + a^2(\sigma^2 - \rho + 2) + 4b^2 = 0,$$
$$2ag(\pi - \sigma\rho) - 4bf(1 - \rho) + c(\pi - \sigma\rho) + a^2(\pi - \sigma\rho) = 0,$$
$$2ag(\sigma\pi - 1) - 4bf\pi + c(\sigma\pi + 1) + a^2(\sigma\pi + 1) = 0.$$

These give the values of f, g, c, where g, f are the coordinates of the circumcentre.

197. *Example* 18.—Find the condition that three points t_1, t_2, t_3 be conormal; and when this condition is satisfied, find the coordinates of the normal point.

(i) Conormal points lie on the hyperbola—

$$(a^2 - b^2)xy + b^2kx - a^2hy = 0.$$

Hence if the normal at t passes through (h, k), t must satisfy—

$$ab(a^2 - b^2)2t(1 - t^2) + ab^2k(1 - t^2)(1 + t^2) - a^2bh2t(1 + t^2) = 0,$$
or $$(a^2 - b^2)2t(t^2 - 1) + bk(t^4 - 1) + 2aht(t^2 + 1) = 0.$$

Since the given t's are the roots of $t^3 - \sigma t^2 + \rho t - \pi = 0$, this equation of degree 4 in t can be reduced to the form $ut^2 + vt + w = 0$; and then we must have $u = 0$, $v = 0$, $w = 0$. These equations are—

$$2(a^2 - b^2)\sigma + 2ah\sigma + bk(\sigma^2 - \rho) = 0,$$
$$-2(a^2 - b^2)(1 + \rho) + 2ah(1 - \rho) + bk(\pi - \sigma\rho) = 0,$$
$$2(a^2 - b^2)\pi + 2ah\pi + bk(\sigma\pi - 1) = 0.$$

Since these are three equations for the determination of two quantities h, k, one condition must be satisfied.

The first and third equations can be written—

$$\sigma[2(a^2 - b^2) + 2ah + bk\sigma] = bk\rho,$$
$$\pi[2(a^2 - b^2) + 2ah + bk\sigma] = bk,$$

hence $\sigma = \pi\rho$ is the condition.

(ii) When this is satisfied, the coordinates of the normal point $x(= h)$ and $y(= k)$ are given by—

$$2ax(1 - \rho) + by(\pi - \sigma\rho) = 2(a^2 - b^2)(1 + \rho),$$
$$2ax\pi + by(\sigma\pi - 1) = - 2(a^2 - b^2)\pi.$$

But $\sigma = \pi\rho$, hence—

$$2ax(1 - \rho) + by\pi(1 - \rho^2) = 2(a^2 - b^2)(1 + \rho),$$
$$-2ax\pi + by(1 - \pi^2\rho) = 2(a^2 - b^2)\pi,$$

from which $$x = \frac{a^2 - b^2}{a} \cdot \frac{1 + \rho}{1 - \rho} \cdot \frac{1 - \pi^2}{1 + \pi^2},$$
$$y = \frac{2(a^2 - b^2)}{b} \cdot \frac{1}{1 - \rho} \cdot \frac{2\pi}{1 + \pi^2}.$$

Note.—Since the parameters of four conormal points satisfy $t_1t_2t_3t_4 = -1$, the fourth point is $-\frac{1}{\pi}$.

198. *Example* 19.—Find the centroid of a circumscribed triangle whose points of contact are conormal; also of the inscribed triangle. Find also in each case the orthocentre and the circumcentre.

(i) A vertex of the circumscribed triangle is—

$$\left(a\frac{1 - t_2t_3}{1 + t_2t_3}, b\frac{t_2 + t_3}{1 + t_2t_3}\right);$$

hence the centroid is given by—

$$\frac{3x}{a} = \Sigma\frac{1 - t_2t_3}{1 + t_2t_3} = \Sigma\frac{t_1 - \pi}{t_1 + \pi}, \quad \frac{3y}{b} = \Sigma\frac{t_2 + t_3}{1 + t_2t_3} = \Sigma\frac{-t_1^2 + t_1\pi\rho}{t_1 + \pi}.$$

The relation that connects t with π, ρ, namely—

$$t^3 - \pi\rho t^2 + \rho t - \pi = 0,$$

makes it possible to express $\dfrac{t-\pi}{t+\pi}$ and $\dfrac{-t^2+t\pi\rho}{t+\pi}$ in the form pt^2+qt+r. Write $\dfrac{t-\pi}{t+\pi} \equiv pt^2+qt+r$,

then $\quad t-\pi \equiv pt^3+(p\pi+q)t^2+(q\pi+r)t+r\pi$,

or $\quad pt^3+(p\pi+q)t^2+(q\pi+r-1)t+(r+1)\pi = 0$,

is satisfied by t_1, t_2, t_3. It must therefore be the same as $t^3 - \pi\rho t^2 + \rho t - \pi = 0$. Hence—

$$p\pi + q = -p\pi\rho,\ q\pi + r - 1 = p\rho,\ (r+1)\pi = -p\pi.$$

These equations give $p = \dfrac{-2}{(1+\rho)(1+\pi^2)}$, $q = \dfrac{2\pi}{1+\pi^2}$, $r = -1 + \dfrac{2}{(1+\rho)(1+\pi^2)}$, and therefore—

$$\frac{t_1-\pi}{t_1+\pi} = \frac{1}{(1+\rho)(1+\pi^2)}[-2t_1^2 + 2\pi(1+\rho)t_1 + 2 - (1+\rho)(1+\pi^2)].$$

Hence—

$$\frac{3x}{a} = \frac{1}{(1+\rho)(1+\pi^2)}[-2\Sigma t_1^2 + 2\pi(1+\rho)\Sigma t_1 + 6 - 3(1+\rho)(1+\pi^2)].$$

Now $\Sigma t_1 = \sigma = \pi\rho$, $\Sigma t_1^2 = (\Sigma t_1)^2 - 2\Sigma t_1 t_2 = \pi^2\rho^2 - 2\rho$, hence—

$$\begin{aligned} x &= \frac{a}{3(1+\rho)(1+\pi^2)}[-2\pi^2\rho^2 + 4\rho + 2\pi^2\rho(1+\rho) + 6 - 3 - 3\pi^2 - 3\rho - 3\pi^2\rho] \\ &= \frac{a}{3(1+\rho)(1+\pi^2)}[3 + \rho - 3\pi^2 - \pi^2\rho] \\ &= \frac{a(3+\rho)(1-\pi^2)}{3(1+\rho)(1+\pi^2)}. \end{aligned}$$

Similarly, if we write $\dfrac{-t^2+\pi\rho t}{t+\pi} \equiv pt^2+qt+r$, we find—

$$p=\frac{-\pi}{1+\pi^2},\ q=\frac{\pi^2\rho-1}{1+\pi^2},\ r=\frac{\pi}{1+\pi^2}.$$

Hence—

$$\begin{aligned} y &= \frac{b}{3(1+\pi^2)}[-\pi\Sigma t_1^2+(\pi^2\rho-1)\Sigma t_1+3\pi] \\ &= \frac{b}{3(1+\pi^2)}[-\pi(\pi^2\rho^2-2\rho)+\pi\rho(\pi^2\rho-1)+3\pi] \\ &= \frac{b(3+\rho)2\pi}{3.2.(1+\pi^2)}. \end{aligned}$$

The centroid of the circumscribed triangle is therefore—

$$\left(\frac{a}{3}\cdot\frac{3+\rho}{1+\rho}\cdot\frac{1-\pi^2}{1+\pi^2},\ \frac{b}{3}\cdot\frac{3+\rho}{2}\cdot\frac{2\pi}{1+\pi^2}\right).$$

(ii) A vertex of the inscribed triangle is the point $\left(a\dfrac{1-t_1^2}{1+t_1^2},\ b\dfrac{2t_1}{1+t_1^2}\right)$. As before, we can express $\dfrac{1-t^2}{1+t^2}$ as pt^2+qt+r. This requires that—

$$pt^4+qt^3+(p+r+1)t^2+qt+r-1=0,$$

or $$(p\pi\rho+q)t^3+(-p\rho+p+r+1)t^2+(p\pi+q)t+r-1=0,$$

be satisfied by $t_1,\ t_2,\ t_3$. Hence—

$$\begin{aligned} -p\rho+p+r+1 &= -\pi\rho(p\pi\rho+q), \\ p\pi+q &= \rho(p\pi\rho+q), \\ r-1 &= -\pi(p\pi\rho+q), \end{aligned}$$

from which $p=\dfrac{-2}{(1-\rho)(1+\pi^2)}$, $q=\dfrac{2(1+\rho)\pi}{(1-\rho)(1+\pi^2)}$,

$$r=1-\frac{2\pi^2}{(1-\rho)(1+\pi^2)}.$$

Hence—

$$x = \frac{a}{3(1-\rho)(1+\pi^2)}[-2\Sigma t_1{}^2 + 2(1+\rho)\pi\Sigma t_1 - 6\pi^2 + 3(1-\rho)(1+\pi^2)]$$

$$= \frac{a}{3(1-\rho)(1+\pi^2)}[-2\pi^2\rho^2 + 4\rho + 2\pi^2\rho(1+\rho) - 6\pi^2 + 3 + 3\pi^2 - 3\rho - 3\pi^2\rho]$$

$$= \frac{a}{3}\cdot\frac{3+\rho}{1-\rho}\cdot\frac{1-\pi^2}{1+\pi^2}.$$

Similarly if we write $\frac{2t}{1+t^2} \equiv pt^2 + qt + r$, we find

$$p = \frac{2\pi}{(1-\rho)(1+\pi^2)},\quad q = \frac{2(1-\pi^2\rho)}{(1-\rho)(1+\pi^2)},$$

$$r = -\frac{2\pi}{(1-\rho)(1+\pi^2)};$$

hence—

$$y = \frac{2b}{3(1-\rho)(1+\pi^2)}[\pi\Sigma t_1{}^2 + (1-\pi^2\rho)\Sigma t_1 - 3\pi]$$

$$= \frac{2b}{3(1-\rho)(1+\pi^2)}[\pi^3\rho^2 - 2\pi\rho + \pi\rho - \pi^3\rho^2 - 3\pi]$$

$$= \frac{-2b(3+\rho)\pi}{3(1-\rho)(1+\pi^2)}.$$

The centroid of the inscribed triangle is therefore—

$$\left(\frac{a}{3}\cdot\frac{3+\rho}{1-\rho}\cdot\frac{1-\pi^2}{1+\pi^2},\quad -\frac{b}{3}\cdot\frac{3+\rho}{1-\rho}\cdot\frac{2\pi}{1+\pi^2}\right).$$

(iii) The equations already found for the orthocentre and circumcentre of two triangles, when simplified by means of the relation $\sigma = \pi\rho$, give the desired results for the conormal triangles.

For the circumscribed triangle at three conormal points, the orthocentre is—

$$\left(\frac{2a^2 + b^2(1+\rho)}{2a}\cdot\frac{1-\pi^2}{1+\pi^2},\ \frac{2a^2 + b^2(1+\rho)}{b(1+\rho)}\cdot\frac{2\pi}{1+\pi^2}\right);$$

the circumcentre is—

$$\left(\frac{4a^2 - b^2(1+\rho)^2}{4a(1+\rho)}\cdot\frac{1-\pi^2}{1+\pi^2},\ -\frac{4a^2 - b^2(1+\rho)^2}{4b(1+\rho)}\cdot\frac{2\pi}{1+\pi^2}\right);$$

the centroid is—

$$\left(\frac{a}{3}\cdot\frac{3+\rho}{1+\rho}\cdot\frac{1-\pi^2}{1+\pi^2},\ \frac{b}{3}\cdot\frac{3+\rho}{2}\cdot\frac{2\pi}{1+\pi^2}\right).$$

For the inscribed conormal triangle, the orthocentre is—

$$\left(\frac{a^2(1+\rho) + 2b^2}{a(1-\rho)}\cdot\frac{1-\pi^2}{1+\pi^2},\ -\frac{a^2(1+\rho) + 2b^2}{b(1-\rho)}\cdot\frac{2\pi}{1+\pi^2}\right);$$

the circumcentre is—

$$\left(\frac{a^2 - b^2}{a(1-\rho)}\cdot\frac{1-\pi^2}{1+\pi^2},\ \frac{(a^2 - b^2)}{2b}\cdot\frac{1+\rho}{1-\rho}\cdot\frac{2\pi}{1+\pi^2}\right);$$

the centroid is—

$$\left(\frac{a}{3}\cdot\frac{3+\rho}{1-\rho}\cdot\frac{1-\pi^2}{1+\pi^2},\ -\frac{b}{3}\cdot\frac{3+\rho}{1-\rho}\cdot\frac{2\pi}{1+\pi^2}\right).$$

The normal point is—

$$\left(\frac{a^2 - b^2}{a}\cdot\frac{1+\rho}{1-\rho}\cdot\frac{1-\pi^2}{1+\pi^2},\ \frac{2(a^2 - b^2)}{b}\cdot\frac{1}{1-\rho}\cdot\frac{2\pi}{1+\pi^2}\right).$$

From the form of the expressions for the coordinates of these seven points it is seen that if ρ has any given value, while π varies, all describe ellipses. If π is given, while ρ varies, the three points t_1, t_2, t_3 are conormal with a fixed point, namely, with $-\dfrac{1}{\pi}$.

199. *Example* 20.—Find all conics that pass through the four points t_1, t_2, t_3, τ, where the three t's are given by $t^3 - \sigma t^2 + \rho t - \pi = 0$. Find the value of τ in order that a circle may pass through the points.

The general conic may be written—

$$\alpha\frac{x^2}{a^2} + 2h\frac{xy}{ab} + \beta\frac{y^2}{b^2} + 2g\frac{x}{a} + 2f\frac{y}{b} + c = 0.$$

Where this meets the ellipse, $\dfrac{x}{a} = \dfrac{1-t^2}{1+t^2}$, $\dfrac{y}{b} = \dfrac{2t}{1+t^2}$; hence the parameters of the four common points are the roots of

$$\alpha(t^2-1)^2 - 2h\,.\,2t(t^2-1) + \beta\,.\,4t^2 - 2g(t^4-1) \\ + 2f\,.\,2t(t^2+1) + c(t^2+1)^2 = 0,$$

that is, of

$$(\alpha - 2g + c)t^4 - (4h - 4f)t^3 + (-2\alpha + 4\beta + 2c)t^2 \\ + (4h + 4f)t + (\alpha + 2g + c) = 0.$$

But the four parameters are given as the roots of

$$t^4 - (\sigma + \tau)t^3 + (\rho + \sigma\tau)t^2 - (\pi + \rho\tau)t + \pi\tau = 0;$$

hence

$$\begin{aligned} 4h - 4f &= (\sigma + \tau)(\alpha - 2g + c), \\ -2\alpha + 4\beta + 2c &= (\rho + \sigma\tau)(\alpha - 2g + c), \\ -4h - 4f &= (\pi + \rho\tau)(\alpha - 2g + c), \\ \alpha + 2g + c &= \pi\tau(\alpha - 2g + c). \end{aligned}$$

The α, β, c enter only in the combinations $\alpha - \beta$, $\alpha + c$; write therefore $\alpha - \beta = 2p$, $\alpha + c = 2k$, then the equations become, in a different order—

$$\begin{aligned} 2h - 2f &= (\sigma + \tau)(k - g), \\ -2h - 2f &= (\pi + \rho\tau)(k - g), \\ 2k - 4p &= (\rho + \sigma\tau)(k - g), \\ k + g &= \pi\tau(k - g). \end{aligned}$$

The last of these gives $k : g = \pi\tau + 1 : \pi\tau - 1$. Since one of the coefficients may be chosen arbitrarily, take $g = \pi\tau - 1$, then $k = \pi\tau + 1$, and $k - g = 2$. Hence the third equation gives

$$\begin{aligned} 2p &= k - (\rho + \sigma\tau) \\ &= \tau(\pi - \sigma) + 1 - \rho. \end{aligned}$$

Also $h - f = \sigma + \tau$, $-h - f = \pi + \rho\tau$, from which

$$\begin{aligned} -2f &= \tau(1 + \rho) + (\pi + \sigma), \\ 2h &= \tau(1 - \rho) - (\pi - \sigma). \end{aligned}$$

The value found for $2p$ gives $\beta = \alpha - \tau(\pi - \sigma) - (1 - \rho)$, and the value for $2k$ gives $c = -\alpha + 2\pi\tau + 2$, hence the equation of the general conic through the four points is—

$$\begin{aligned} &\alpha\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) \\ &\quad - \left[(\pi - \sigma)\frac{xy}{ab} + (1 - \rho)\frac{y^2}{b^2} + 2\frac{x}{a} + (\pi + \sigma)\frac{y}{b} - 2\right] \\ &\quad + \tau\left[(1 - \rho)\frac{xy}{ab} - (\pi - \sigma)\frac{y^2}{b^2} + 2\pi\frac{x}{a} - (1 + \rho)\frac{y}{b} + 2\pi\right] = 0. \end{aligned}$$

If τ is given, this equation involves the single parameter α, and the conics (through four given points) form a pencil. If τ is not given, the equation involves *two* parameters α, τ, and represents all conics through the three points t_1, t_2, t_3. In particular, the circle through these three points is obtained by making—

$$\begin{aligned} \frac{\alpha}{a^2} &= \frac{\alpha}{b^2} - \frac{1 - \rho}{b^2} - \frac{\tau(\pi - \sigma)}{b^2}, \\ \frac{\pi - \sigma}{ab} &- \frac{\tau(1 - \rho)}{ab} = 0. \end{aligned}$$

The second of these equations gives $\tau = \dfrac{\pi - \sigma}{1 - \rho}$, and the first then gives $\alpha = \dfrac{a^2}{a^2 - b^2} \cdot \dfrac{(1 - \rho)^2 + (\pi - \sigma)^2}{1 - \rho}$; hence the circle through the three points t_1, t_2, t_3 is—

$$x^2 + y^2 - 2\frac{a^2 - b^2}{a} \cdot \frac{1 - \rho - \pi^2 + \pi\sigma}{(1 - \rho)^2 + (\pi - \sigma)^2}x$$
$$- 2\frac{a^2 - b^2}{b} \frac{\pi - \sigma\rho}{(1 - \rho)^2 + (\pi - \sigma)^2}y$$
$$- a^2 + \frac{2(a^2 - b^2)(1 - \rho + \pi^2 - \pi\sigma)}{(1 - \rho)^2 + (\pi - \sigma)^2} = 0.$$

If the points are conormal, this reduces to—

$$x^2 + y^2 - 2\frac{a^2 - b^2}{a} \frac{1}{1 - \rho} \cdot \frac{1 - \pi^2}{1 + \pi^2} \cdot x$$
$$- 2\frac{a^2 - b^2}{b} \cdot \frac{1 + \rho}{1 - \rho} \cdot \frac{\pi}{1 + \pi^2} \cdot y + a^2\frac{1 + \rho}{1 - \rho} - \frac{2b^2}{1 - \rho} = 0.$$

EXAMPLES.

1. Prove that if t_1t_2 is constant, the chord passes through a fixed point on the axis of x. Find the value of t_1t_2 if the fixed point is a focus.

2. The orthocentre of a circumscribed triangle lies at a focus. Prove that the vertices lie on a circle of radius $2a$ whose centre is the other focus.

Find the locus of the orthocentre of the corresponding inscribed triangle.

3. Find the locus of the orthocentre of the triangle formed by two fixed tangents and one variable tangent.

4. Find the locus of the orthocentre of an inscribed triangle, two of whose vertices are fixed.

5. Two sides of an inscribed triangle pass through the foci; find the envelope of the third side.

6. Two vertices of a circumscribed triangle describe the directrices; find the locus of the third vertex.

7. Show that the four points whose parameters satisfy

$$t^2 - \lambda t + \mu = 0,\ t^2 - \lambda' t + \mu' = 0,$$

are conormal if $\mu\mu' = -1$, $\lambda\lambda' = -(\mu + \mu')$.

8. Prove that if the orthocentre of the triangle circumscribed at PQR is at the centre of the ellipse, the points P, Q, R are conormal. What is the value of ρ?

Find the locus of the normal point.

9. Prove that if the orthocentre of an inscribed triangle is at the centre of the ellipse, the triangle is conormal. Find the locus of the normal point.

10. Prove that if the orthocentre of an inscribed conormal triangle is at the normal point, the circumcentre of the circumscribed triangle is also at the normal point, and the two centroids are at the centre of the ellipse.

11. Prove that if a chord passes through a fixed point on the axis of x, the conormal chord also passes through a fixed point on the axis of x.

12. A chord passes through a focus; prove that the conormal chord passes through the foot of the other directrix.

13. Find the locus of the intersection of conormal chords through fixed points on the axis of x.

14. A chord passes through a fixed point (h, k); find the locus of the intersection with the conormal chord.

15. If the circumcentre of the circumscribed triangle is at the centre of the ellipse the three points of contact of the sides are conormal; the radius of the circumcircle is $a \pm b$; the normal point, and the orthocentre of the circumscribed triangle, describe a circle of radius $a \mp b$, on which they are at opposite extremities of a diameter.

16. Prove that the sides of all conormal triangles for which ρ has a given value are tangents to a certain ellipse, and that the vertices of the corresponding circumscribed triangles lie on another ellipse.

17. Show that the only circles with C for centre that have the property that triangles can be inscribed to the circle and circumscribed to the ellipse are of radius $a \pm b$. Show that the points of contact of such a triangle with the ellipse are conormal, and have a constant ρ.

18. Prove that the points whose parameters are the roots of

$t^4 - \lambda t^3 + \mu t^2 - \nu t + \kappa = 0$ are conormal if $\mu = 0$, $\kappa = -1$; and that the normal point is $\left(\frac{a^2-b^2}{a} \cdot \frac{\lambda+\nu}{\lambda-\nu}, \; -\frac{a^2-b^2}{b} \cdot \frac{4}{\lambda-\nu}\right)$.

19. Prove that if four points P_1, P_2, P_3, P_4 are concyclic, then D_1, D_2, D_3, D_4, the corresponding extremities of diameters conjugate to CP_1, CP_2, CP_3, CP_4, are also concyclic. Prove also that the coordinates of the centre of the circle $D_1D_2D_3D_4$ are related to the coordinates of the centre of the circle $P_1P_2P_3P_4$ precisely as the coordinates of any D to the coordinates of the corresponding P.

20. Prove that the fourth point at which the circle through three conormal points again meets the ellipse, and the fourth point conormal with the three, are at opposite extremities of a diameter of the ellipse.

21. Prove that if the normals at P, Q, R meet on the normal at Z, the circle through P, Q, R passes through two fixed points on the tangent at Z′, the other extremity of the diameter through Z, namely, Z′ itself, and the foot of the perpendicular from the centre of the ellipse.

22. If the normals at P, Q, R meet on the normal at Z, then for all possible positions of the triangle PQR the sides are tangents to a certain parabola; and the vertices of the circumscribed triangle lie on a certain rectangular hyperbola, whose centre is on the ellipse, at Z′.

23. If the normal point for a conormal triangle describes the ellipse, find the value of ρ; show that the orthocentre of the circumscribed triangle and the centroid of the inscribed triangle describe ellipses similar to the given one, and similarly placed; the orthocentre of the inscribed triangle and the centroid of the circumscribed triangle describe ellipses similar to the given one, but differently placed. Find also the envelope of the sides of the inscribed triangle.

24. If $e^2 = \frac{4}{5}$, the circumcentre and the centroid of the circumscribed triangle at three conormal points, for which the normal point is on the ellipse, have the same locus, and the line that joins them is parallel to the axis of x.

If $e^2 = \frac{2}{7}$, the orthocentre of the circumscribed triangle and the centroid of the inscribed triangle have the same locus, and the line that joins them is parallel to the axis of x.

25. Find the condition that the three points t_1, t_2, t_3 on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be conormal. Find the orthocentre, circumcentre, and centroid of the circumscribed and inscribed triangles, and the normal point.

Note.—When the coordinates of these points have been found, results analogous to those here stated for the ellipse can readily be investigated.

26. Prove, for the rectangular hyperbola, that if ρ is given, the centroid of an inscribed conormal triangle and the circumcentre of the corresponding circumscribed triangle describe rectangular hyperbolas.

27. Find the orthocentre, circumcentre, and centroid of the inscribed and circumscribed conormal triangles for a rectangular hyperbola when $\rho = -3$. Find also the normal point. Represent on a diagram, taking any point as π.

28. If the coordinates of a point on $xy = a^2$ are expressed as $\left(ma, \frac{a}{m}\right)$, form the equation whose roots are the parameters of the vertices of an equilateral triangle.

29. Find the relation satisfied by the parameters of four concyclic points on $xy = a^2$.

30. Find the relation satisfied by the parameters of three conormal points on $xy = a^2$.

31. Find the two relations satisfied by the parameters of four conormal points on $xy = a^2$.

32. Show that the orthocentre of the triangle whose vertices have for parameters the roots of $m^3 - \sigma m^2 + \rho m - \pi = 0$ is the point whose parameter is $-\frac{1}{\pi}$.

Chord of given length.

200. *Example* 21.—Find the equation of a chord of a parabola, of slope m, length $2l$.

The chord $y = mx + n$ meets the parabola $y^2 = 4px$ where $m^2x^2 + 2(mn - 2p)x + n^2 = 0$. If the inclination is θ,

$$2l \cos \theta = x_1 - x_2,$$

where $$\cos\theta = \frac{1}{\sqrt{m^2+1}},$$

and $$(x_1 - x_2)^2 = \frac{4(mn-2p)^2}{m^4} - \frac{4n^2}{m^2}$$
$$= \frac{16p(p-mn)}{m^4}.$$

Hence $$\frac{4l^2}{1+m^2} = \frac{16p(p-mn)}{m^4},$$

from which $$n = \frac{p}{m} - \frac{l^2m^3}{4p(m^2+1)}.$$

The equation of the chord is therefore—

$$y = mx + \frac{p}{m} - \frac{l^2m^3}{4p(m^2+1)}.$$

Equilateral Triangle.

201. *Example* 22.—Find an equilateral triangle inscribed to a central conic.

If the circle through PQR meets the conic again at T, the angles that an axis makes with TP, TQ, TR are supplementary to the angles that it makes with QR, RP, PQ. Hence if PQR is equilateral, the lines TP, TQ, TR must be inclined at angles 60°, and therefore if T is taken as origin, the equation of the three lines must be—

$$x^3 - 3xy^2 = \lambda(3x^2y - y^3).$$

If T is (h, k), the equation of the conic $\frac{x^2}{\alpha} + \frac{y^2}{\beta} = 1$, when the origin is transferred to T, becomes—

$$\frac{x^2}{\alpha} + \frac{y^2}{\beta} + \frac{2hx}{\alpha} + \frac{2ky}{\beta} = 0,$$

and the equation of a circle through (h, k)—

$$(x-p)^2+(y-q)^2=r^2,$$

becomes $$x^2+y^2+2(h-p)x+2(k-q)y=0.$$

The lines that join the origin to the remaining three intersections of the ellipse and circle are found by combining these two equations so as to obtain a homogeneous equation. Hence they are—

$$(x^2+y^2)\left(\frac{hx}{\alpha}+\frac{ky}{\beta}\right)=\left(\frac{x^2}{\alpha}+\frac{y^2}{\beta}\right)[(h-p)x+(k-q)y].$$

This equation is of the desired form, $x^3-3xy^2=\lambda(3x^2y-y^3)$, if $3\times$ coef. x^3+ coef. $xy^2=0$, $3\times$ coef. y^3+ coef. $x^2y=0$,

that is, if $$3\left[\frac{h}{\alpha}-\frac{h-p}{\alpha}\right]+\frac{h}{\alpha}-\frac{h-p}{\beta}=0,$$

and $$3\left[\frac{k}{\beta}-\frac{k-q}{\beta}\right]+\frac{k}{\beta}-\frac{k-q}{\alpha}=0.$$

Hence $$p(\alpha+3\beta)=h(\alpha-\beta),$$
$$q(3\alpha+\beta)=-k(\alpha-\beta).$$

The radius of the circle, r, is determined by—

$$r^2=(h-p)^2+(k-q)^2$$
$$=16\left[\frac{h^2\beta^2}{(\alpha+3\beta)^2}+\frac{k^2\alpha^2}{(3\alpha+\beta)^2}\right].$$

Hence we may take for (h, k) any point on the ellipse, and then the circle is uniquely determined.

EXAMPLES.

1. Find the line-equation of the envelope of a chord of a parabola, of length $2l$.

2. Find the locus of the point of bisection of a chord of a parabola, of length $2l$.

3. Prove that four chords of a parabola (real or imaginary) of any specified length pass through a given point.

4. Find the locus of the foot of the perpendicular from the vertex of a parabola to a chord of given length.

5. Find the locus of the foot of the perpendicular from the focus of a parabola to a chord of given length.

6. Find the locus of the pole, with respect to a parabola, of a chord of given length.

7. Show that the equation of a chord of a central conic, of slope m and length $2l$, is $y = mx + n$, where $n^2 = am^2 + \beta - \dfrac{l^2(am^2 + \beta)^2}{a\beta(m^2 + 1)}$.

8–12. Apply Nos. 1, 2, 3, 5, 6 to central conics.

13. Find the locus of a point if two of the chords of length $2l$ that pass through it are at right angles (for parabola and for central conics).

14. Prove that the circle described with centre at any point of a rectangular hyperbola to pass through the other extremity of the diameter cuts the hyperbola at the vertices of an equilateral triangle.

15. Show that for all values of h the points on $xy = a^2$ whose abscissæ are the roots of $h^2x^3 - 3h^3x^2 - 3a^4x + a^4h = 0$ are the vertices of an equilateral triangle.

16. Find equilateral triangles inscribed to a parabola.

17. Find the locus of the centre of an equilateral triangle inscribed to—(1) a parabola, (2) a central conic.

18. Prove that the circumcircles of equilateral triangles circumscribed to a parabola form a coaxal system.

19. Prove that the vertices of equilateral triangles circumscribed to a parabola lie on the hyperbola $3x^2 - y^2 + 10px + 3p^2 = 0$.

20. Prove that for any value of λ the two parabolas—

$$x^2 + 3px - \lambda y = 0, \ y^2 - px - 3\lambda y - 3p^2 = 0,$$

which meet at $(-3p, 0)$, meet also at the vertices of an equilateral triangle circumscribed to the parabola $y^2 = 4px$.

21. The tangents at the extremities of a focal chord of a central conic meet at P, and the normals at Q. Show that PQ passes through the other focus.

22. Find the condition that the normal to $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ at (x_1, y_1) hall be tangent to $-\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at (x_2, y_2). Prove that the normal

to the second hyperbola at (x_2, y_2) is then the tangent to the first hyperbola at $(-x_1, -y_1)$.

Find the points for the hyperbolas $\frac{x^2}{4} - \frac{y^2}{3} = \pm 1$.

23. Prove that the locus of a point whose polar with respect to a parabola touches a given circle, whose centre is on the axis and whose diameter is equal to the latus rectum of the parabola, is a rectangular hyperbola whose axis major is equal to the latus rectum of the parabola.

24. Find the conditions that $\xi x + \eta y + 1 = 0$ be an asymptote of the general conic.

25. Find the conditions that the line (ξ, η) be an asymptote of a conic $u + \lambda v = 0$, where—

$$u = ax^2 + 2hxy + by^2 + 2gx + 2fy + c,$$
$$v = a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c'.$$

Hence find the line-equation of the envelope of the asymptotes of a pencil of conics. How many asymptotes pass through a given point?

26. Show that $y = mx + n$ is an asymptote of $u + \lambda v = 0$, where

$$u = ax^2 + 2hxy + by^2 + 2gx + 2fy + c,$$
$$v = 2xy + c',$$

if $$n = \frac{2m(fm+g)}{a - bm^2}, \text{ and that then } \lambda = -\frac{a + bm^2}{m}.$$

27. Form the equation for the slopes of the three asymptotes through a given point to curves of the pencil of Ex. 26. Find the point for which the three asymptotes are equally inclined.

28. Prove that the locus of a point, such that two of the asymptotes from it to conics of a pencil are perpendicular, is a hyperbola unless the pencil contains a circle, and then it is a straight line perpendicular to the line that joins the centre of the circle to the centre of the rectangular hyperbola of the pencil.

29. Prove that the third asymptote through the centre (x_1, y_1) of a conic of the pencil of Ex. 26 is $\frac{x}{x_1} + \frac{y}{y_1} = 2$.

Exs. 30-36 relate to a pencil of rectangular hyperbolas.

30. Prove that the centre-locus is a circle.

31. Prove that hyperbolas whose centres are at diametrically opposite points on the centre-locus have the axes of one parallel to the asymptotes of the other. [Call these opposite hyperbolas.]

32. Find the conditions that $y = mx + n$ be an asymptote to a curve of the pencil. Form the equation whose roots give the slopes of the asymptotes through a point. Hence show that at least one real curve of the pencil has an asymptote through the centre of the centre-locus.

33. Show that by proper choice of axes the equation of the pencil can be reduced to $u + \lambda v = 0$, where

$$u = x^2 - y^2 - 2ax + c,$$
$$v = 2xy + 2ay + c';$$

and that for opposite hyperbolas $\lambda_1\lambda_2 = -1$.

34. Prove that the polars of any point on the centre-locus with respect to all the hyperbolas are parallel.

35. Find the locus of a point whose polars with respect to a given pair of opposite hyperbolas are perpendicular. How is the locus related to the two hyperbolas?

36. Show that the locus of a point whose polars with respect to a given pair of opposite hyperbolas make a constant angle is a circle; and that the circles obtained for all values of the angle form a coaxal system, whose radical axis passes through the centres of the two hyperbolas.

The Isoptic Locus.

202. *Example* 23.—Find the locus of the point of intersection of tangents to a central conic that make a given angle θ, where $\tan\theta = k$.

The points of contact of the tangents from a point P are seen as the extreme points of the conic when viewed from P; the conic lies entirely in one pair of the angles formed by the tangents. Hence in the problem proposed the conic subtends a constant angle at P; to an eye placed at the various positions of P, it appears to occupy a constant angle. The locus of P is therefore called an isoptic locus. If the angle is a right angle, the locus is called the orthoptic locus.

The equation which gives the slopes of the tangents from (x', y') is $(x'^2 - \alpha)m^2 - 2x'y'm + y'^2 - \beta = 0$. The roots of this equation are connected by the relation which expresses the given condition, namely, that the angle between the tangents has an assigned magnitude; this relation is $\pm \dfrac{m_1 - m_2}{1 + m_1 m_2} = k$.

Hence—

$$\pm\sqrt{\frac{4x'^2y'^2}{(x'^2 - \alpha)^2} - \frac{4(y'^2 - \beta)}{x'^2 - \alpha}} = k\left(1 + \frac{y'^2 - \beta}{x'^2 - \alpha}\right).$$

The equation of the isoptic locus is therefore—

$$\pm 2\sqrt{\beta x^2 + \alpha y^2 - \alpha\beta} = k(x^2 + y^2 - \alpha - \beta),$$

that is—

$$\tan^2\theta(x^2 + y^2 - \alpha - \beta)^2 - 4(\beta x^2 + \alpha y^2 - \alpha\beta) = 0.$$

This is—

$$(x^2 + y^2)^2 - 2(\alpha + \beta)(x^2 + y^2) - 4\cot^2\theta(\beta x^2 + \alpha y^2) + (\alpha + \beta)^2 + 4\alpha\beta\cot^2\theta = 0.$$

This may be written—

$$(x^2 + y^2)^2 - 2gx^2 - 2fy^2 + c = 0,$$

where

$$\alpha + \beta + 2\beta\cot^2\theta = g,$$
$$\alpha + \beta + 2\alpha\cot^2\theta = f,$$
$$(\alpha + \beta)^2 + 4\alpha\beta\cot^2\theta = c.$$

Note that if the conic is not given, but only an isoptic locus, so that g, f, c are given, these equations determine α, β, and θ.

Example 24.—Show that two conics have a given isoptic locus, $(x^2 + y^2)^2 - 2gx^2 - 2fy^2 + c = 0$.

The equations for α, β, θ are—

$$\alpha + \beta + 2\beta \cot^2 \theta = g,$$
$$\alpha + \beta + 2\alpha \cot^2 \theta = f,$$
$$(\alpha + \beta)^2 + 4\alpha\beta \cot^2 \theta = c.$$

Eliminating θ, we obtain—

$$\alpha g + \beta f = c,$$
$$\alpha g - \beta f = \alpha^2 - \beta^2.$$

From these, by eliminating β, we find the equation for α,

$$(f^2 - g^2)\alpha^2 + 2(c - f^2)g\alpha - c^2 + cf^2 = 0.$$

This shows that for a given isoptic locus there are two admissible values for α, such that $\alpha_1\alpha_2 = \dfrac{cf^2 - c^2}{f^2 - g^2}$; and the equation $\alpha g + \beta f = c$ shows that each value of α determines the corresponding β uniquely. Hence there are two conics, both real or both imaginary.

To compare the conics, write $\alpha = \lambda\beta$. The equations for α, β become—

$$\beta(\lambda g + f) = c,$$
$$\beta^2(\lambda^2 - 1) = \beta(\lambda g - f),$$

from which $\quad c(\lambda^2 - 1) = (\lambda g + f)(\lambda g - f).$

Hence $\quad \lambda^2(g^2 - c) = f^2 - c, \lambda = \pm \sqrt{\dfrac{f^2 - c}{g^2 - c}}.$

That is, $\alpha : \beta$ is positive for one conic (if real), negative for the other. If the conics are real, they are, therefore, one an ellipse, the other a hyperbola.

The angle θ, belonging to either conic, is determined from the equations—

$$\alpha + \beta + 2\beta \cot^2 \theta = g,$$
$$\alpha + \beta + 2\alpha \cot^2 \theta = f;$$

or
$$\beta(\lambda + 1 + 2\cot^2 \theta) = g,$$
$$\beta(\lambda + 1 + 2\lambda \cot^2 \theta) = f.$$

Hence
$$g(\lambda + 1 + 2\lambda \cot^2 \theta) = f(\lambda + 1 + 2\cot^2 \theta),$$

from which
$$\cot^2 \theta = \frac{(\lambda + 1)(f - g)}{2(\lambda g - f)},$$

where λ has one of the two values already found.

Even though a conic is real, an isoptic locus may belong to it only in virtue of an imaginary angle.

Note.—If θ is a right angle, the equation of the isoptic (orthoptic) locus reduces to $x^2 + y^2 = \alpha + \beta$. If $f = g$, the locus breaks up into two circles; for the equation is $(x^2 + y^2)^2 - 2g(x^2 + y^2) + c = 0$. The value of λ is 1, the conic is a circle; every isoptic locus of a circle is a circle, or rather, taking into account supplementary values of the angle θ, a pair of circles. The quadratic for α reduces to a simple equation. Similarly if $f = -g$, there is only one value for α; the conic is a rectangular hyperbola.

EXAMPLES.

1. Chords subtend a constant angle at the vertex of a parabola. Find the line-equation and the point-equation of the envelope.

2. Chords subtend a constant angle at the focus of a parabola. Prove that the envelope is an ellipse with the same focus and directrix as the parabola. Express the eccentricity of this ellipse in terms of the angle subtended.

3. Find the envelope of a chord of a rectangular hyperbola that subtends a right angle at the focus.

4. Chords of the conic $x^2 + y^2 = e^2(x + 2p)^2$ subtend at the focus an angle α; prove that the envelope is—

$$x^2 + y^2 = e^2 \sin^2 \frac{\alpha}{2}(x + 2p)^2, \text{ or } x^2 + y^2 = e^2 \cos^2 \frac{\alpha}{2}(x + 2p)^2,$$

according as α is obtuse or acute.

5. Prove that any hyperbola described with the same focus and directrix as a given parabola is an isoptic locus for the parabola; and that any ellipse with the same focus and directrix is the envelope of a chord of the parabola that subtends a constant angle at the focus.

6. A hyperbola is the isoptic locus (with angle α) for a parabola; prove that the parabola is the envelope of chords of the hyperbola that subtend a constant angle at the focus.

7. Two vertices of a triangle circumscribed to a parabola describe hyperbolas with the same focus and directrix as the parabola. Prove that the third vertex describes another hyperbola with the same focus and directrix; and that the triangle is of constant shape.

8. A triangle inscribed in a given conic moves so that two sides are tangents each to one of two conics with the same focus and directrix as the given conic. Prove that the third side in all its position is tangent to a conic.

9. A number of conics have the same focus and directrix, and their eccentricities are in geometric progression. Prove that the part of any tangent to any conic of the system that is included by the *next* conic of the system subtends the same angle at the focus.

10. Two central conics have a common isoptic locus; prove that the sum of the squares of their eccentricities $= 2$.

11. An ellipse and hyperbola have the same transverse and conjugate axes, and the sum of the squares of their eccentricities $= 2$. Prove that they have one common isoptic locus, and find it. Express the isoptic angle for each of the conics in terms of the semi-axes of the two conics.

12. Find the common isoptic locus for two confocal conics, and show that it breaks up into the four isotropic lines through the two foci.

The Graphical Solution of Equations.

203. Among the many applications of analytical geometry one of the most interesting, historically, is the graphical solution or *construction* of equations. Any equation in x combined with an arbitrary equation in x, y gives rise to other equations in x, y. These represent curves; the abscissæ of the points of intersection of any two give the solutions of the equation.

Example— $x^3 - 17x + 20 = 0.$

Combine with $y = x^2,$

then $xy - 17x + 20 = 0.$

Also, if we introduce a factor x, the equation becomes—

$$x^4 - 17x^2 + 20x = 0,$$

which can be written in any of the forms—

$$y^2 - 17y + 20x = 0,$$
$$y^2 - 17x^2 + 20x = 0,$$
$$x^2 + y^2 + 20x - 18y = 0.$$

Thus the roots of the given equation are the abscissæ of the three common (finite) points of the parabola $y = x^2$ and the rectangular hyperbola $xy - 17x + 20 = 0$; they are also the abscissæ of the three intersections, other than the origin, of the parabola and the circle—

$$x^2 + y^2 + 20x - 18y = 0.$$

The value of this as a practical process depends on the facility with which the necessary curves can be constructed. A circle can always be traced, and instruments can be made for the description of the conics. A single parabola $y = x^2$ is however all that is needed for the cubic and biquadratic equations, while for higher equations conics alone are not sufficient. The proof that the roots of an equation of the third or fourth degree can be constructed by means of a fixed parabola and a variable circle will now be given.

The parabola $y = x^2$ meets the circle—

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$

where $$x^2 + x^4 + 2gx + 2fx^2 + c = 0,$$
that is, where $$x^4 + (1 + 2f)x^2 + 2gx + c = 0;$$

hence the abscissæ are the roots of an equation of the fourth degree without the second term; and if $c = 0$, the equation reduces to a cubic without the second term. That is, the roots of the equation—

$$x^4 + qx^2 + rx + s = 0,$$

are the abscissæ of the common points of the parabola—

$$y = x^2$$

and the circle $$x^2 + y^2 + rx + (q - 1)y + s = 0;$$
the roots of $$x^3 + qx + r = 0$$

are the abscissæ of the common points, other than the origin, of the parabola and the circle—

$$x^2 + y^2 + rx + (q - 1)y = 0.$$

This same parabola may be used in a different position. Its equation is—

$$y = x^2 + hx + k,$$

and the abscissæ of its intersections with the circle are the roots of

$$x^2 + (x^2 + hx + k)^2 + 2gx + 2f(x^2 + hx + k) + c = 0,$$

that is, of

$$x^4 + 2hx^3 + (h^2 + 2k + 2f + 1)x^2 + (2hk + 2g + 2fh)x + k^2 + 2fk + c = 0.$$

This can be made to agree with any given quartic equation—

$$x^4 + px^3 + qx^2 + rx + s = 0,$$

by taking

$$\begin{aligned} 2h &= p, \\ h^2 + 2k + 2f + 1 &= q, \\ 2hk + 2g + 2fh &= r, \\ k^2 + 2fk + c &= s. \end{aligned}$$

If we give to k any arbitrary value, we have the right number of equations for the determination of h, f, g, c. Take therefore $k = 0$; then $h = \frac{p}{2}$,

$$\begin{aligned} 2f &= q - 1 - \frac{p^2}{4}, \\ 2g &= r - \frac{p}{2}\left(q - 1 - \frac{p^2}{4}\right), \\ c &= s. \end{aligned}$$

Hence the roots of the given equation are the abscissæ of the intersections of the parabola—

$$y = x^2 + \frac{p}{2}x$$

and the circle—

$$x^2 + y^2 + \left[r - \frac{p}{2}(q-1) + \frac{p^3}{4}\right]x + \left(q - 1 - \frac{p^2}{4}\right)y + s = 0.$$

This shows that we can always use a parabola of given size, with a circle which depends on the given equation. But it is simpler to begin by depriving the equation of the second term, and then the parabola is not only of given size, but is also in a fixed position.

Other conics through the four points of intersection may

be used if preferred. For instance, any two conics of the pencil—

$$x^2 + y^2 + rx + (q - 1)y + s + \lambda(x^2 - y) = 0,$$

will give the roots of

$$x^4 + qx^2 + rx + s = 0;$$

$\lambda = -1$ gives $y^2 + rx + qy + s = 0$, a parabola;

$\lambda = -2$ gives $-x^2 + y^2 + rx + (q + 1)y + s = 0$, a rectangular hyperbola.

204. *Example* 25.—Trisect a given angle.

With the vertex of the angle as centre describe a circle

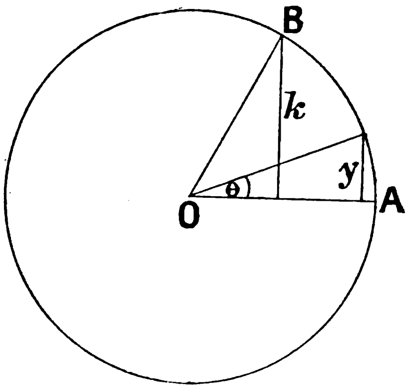

Fig. 97

of radius unity. Then (Fig. 97) $k = \sin a, y = \sin \theta$, where $3\theta = a$.

Now $\qquad \sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta,$

hence $\qquad k = 3y - 4y^3,$

that is, $4y^3 - 3y + k = 0.$

The problem has led to a cubic equation, hence it can be solved graphically by means of conics. By the proof already given for the construction of a cubic equation, the parabola $y^2 = x$ and the circle $x^2 + y^2 - \frac{7}{4}x + \frac{k}{4}y = 0$ will serve. These intersect at three points other than the origin; lines drawn parallel to Ox through these three points meet the circle $x^2 + y^2 = 1$ at points whose angular distances from A are $\frac{a}{3}$, $\frac{2\pi + a}{3}$, $\frac{4\pi + a}{3}$.

The objection to this solution is that instead of using the given circle, we introduce another. It will be more satisfactory to find a conic that cuts the *given* circle at the desired points. For the sake of the construction, it is better to find the second point of trisection, P, so that AOP $= \frac{2}{3}a$. The ordinate of P is therefore a root of

$$4y^3 - 3y + h = 0,$$

where $h = \sin 2a$. The other roots of this equation are $\sin \frac{2}{3}(2\pi + a)$, $\sin \frac{2}{3}(4\pi + a)$, for since

$$\sin a = \sin (2\pi + a) = \sin (4\pi + a),$$

the same equation is obtained for $\sin \frac{2}{3}(2\pi + a)$ and $\sin \frac{2}{3}(4\pi + a)$. The roots of the equation are therefore the ordinates of the vertices of the equilateral triangle PQR (Fig. 98).

We already know that if a circle cuts a rectangular hyperbola at the vertices of an equilateral triangle, the radius of the circle is a diameter of the hyperbola (Ex. 14, § 201); we try therefore to find a rectangular hyperbola with OA as a diameter, that shall meet the circle at points

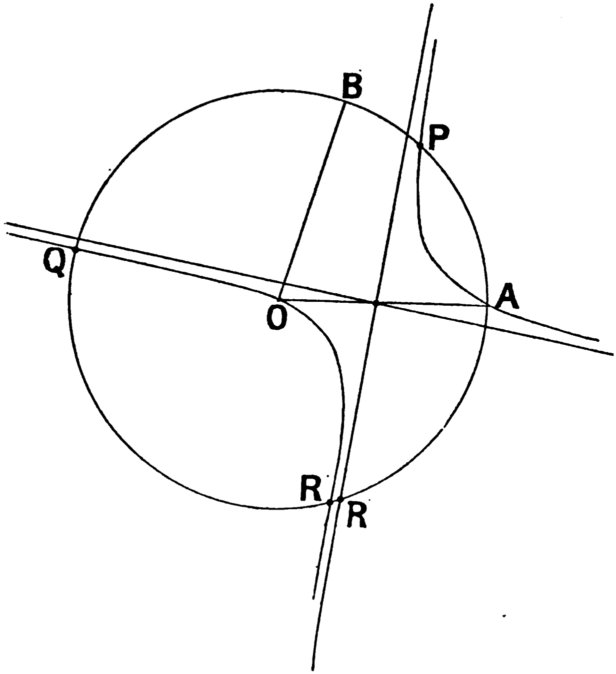

Fig. 98.

whose ordinates are $y = 0$ and the roots of the equation $4y^3 - 3y + h = 0$, that is, at points given by—

$$4y^4 - 3y^2 + hy = 0.$$

A rectangular hyperbola with OA as diameter passes through O and has its centre at $\left(\frac{1}{2}, 0\right)$; its equation is therefore—

$$x^2 + 2mxy - y^2 + 2gx + 2fy = 0,$$

where $x + my + g = 0$, $mx - y + f = 0$ are satisfied by $\left(\frac{1}{2}, 0\right)$. Hence $2g = -1$, $2f = -m$, and the equation is of the form—

$$x^2 + 2mxy - y^2 - x - my = 0.$$

This meets $x^2 + y^2 = 1$ where

$$\begin{aligned} 1 - 2y^2 + 2mxy - x - my &= 0, \\ x(2my - 1) &= 2y^2 + my - 1, \\ x^2(2my - 1)^2 &= (2y^2 + my - 1)^2, \\ (y^2 - 1)(2my - 1)^2 + (2y^2 + my - 1)^2 &= 0, \\ 4(m^2 + 1)y^4 - 3(m^2 + 1)y^2 + 2my &= 0, \\ 4y^4 - 3y^2 + \frac{2m}{m^2 + 1}y &= 0. \end{aligned}$$

Hence $\frac{2m}{m^2 + 1} = h = \sin 2a = \frac{2 \tan a}{\tan^2 a + 1}$, which shows that $m = \tan a$. The desired rectangular hyperbola is therefore

$$x^2 + 2 \tan a . xy - y^2 - x - y \tan a = 0 \text{ (Fig. 98).}$$

205. *Example* 26.—Find conics that shall meet a given circle at four vertices of a regular heptagon.

A side of the heptagon subtends at the centre of the circle an angle a, where $7a = 2\pi$.

Hence $$\sin 3\alpha = -\sin 4\alpha,$$
from which—

$$3 \sin \alpha - 4 \sin^3 \alpha = -4 \sin \alpha \cos\alpha (2 \cos^2 \alpha - 1),$$
that is, $$\sin \alpha[8 \cos^3 \alpha + 4 \cos^2 \alpha - 4 \cos \alpha - 1] = 0.$$

Write $x = \cos \alpha$; the problem is now to find a conic that shall meet the circle $x^2 + y^2 = 1$ where

$$8x^3 + 4x^2 - 4x - 1 = 0,$$

or else where $\sin \alpha = 0$, that is, $\alpha = 0$, and therefore $x = 1$. The equation for the four abscissæ is therefore—

$$(x - 1)(8x^3 + 4x^2 - 4x - 1) = 0,$$
that is, $$8x^4 - 4x^3 - 8x^2 + 3x + 1 = 0.$$

If the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ meets the circle $x^2 + y^2 = 1$ at four points whose abscissæ satisfy this equation, the same is true for every conic of the pencil—

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c + \lambda(x^2 + y^2 - 1) = 0;$$

hence the conic cannot be absolutely determined by the four intersections; another condition is necessary. We may, for instance, choose to find the rectangular hyperbola through the points, and thus take $b = -a$. The conic is now—

$$ax^2 + 2hxy - ay^2 + 2gx + 2fy + c = 0;$$
this meets $$x^2 + y^2 - 1 = 0$$
where $$2ax^2 + 2hxy + 2gx + 2fy + c - a = 0,$$
from which $$2ax^2 + 2gx + c - a = -y(2hx + 2f),$$
hence $$\begin{aligned}(2ax^2 + 2gx + c - a)^2 &= y^2(2hx + 2f)^2 \\ &= (1 - x^2)(2hx + 2f)^2.\end{aligned}$$

The equation for the abscissæ of the four common points is therefore—

$$\begin{aligned}&(4h^2 + 4a^2)x^4 + (8hf + 8ag)x^3 \\ &\quad + [4f^2 - 4h^2 + 4g^2 + 4a(c - a)]x^2 \\ &\qquad + [-8hf + 4g(c - a)]x - 4f^2 + (c - a)^2 = 0,\end{aligned}$$

and this is to be the same as—

$$8x^4 - 4x^3 - 8x^2 + 3x + 1 = 0.$$

Hence the coefficients are proportional, which gives—

$$\begin{aligned}\frac{4h^2 + 4a^2}{8} &= \frac{8hf + 8ag}{-4} = \frac{4f^2 - 4h^2 + 4g^2 + 4a(c - a)}{-8} \\ &= \frac{-8hf + 4g(c - a)}{3} = \frac{-4f^2 + (c - a)^2}{1}.\end{aligned}$$

These are four equations for the determination of the ratios of the five quantities a, c, f, g, h; they determine these ratios, but not uniquely. For the conic may pass (Fig. 99) through either of the two points 1, 6, through either of the two 2, 5, and through either 3 or 4; hence eight ($= 2 \times 2 \times 2$) solutions are to be expected. *One* of these will suffice for the present purpose.

If λ is written for each of the fractions, the equations become—

$$4h^2 + 4a^2 = 8\lambda \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$8hf + 8ag = -4\lambda \quad . \quad . \quad . \quad . \quad (2)$$

$$4f^2 - 4h^2 + 4g^2 + 4a(c - a) = -8\lambda \quad . \quad . \quad . \quad . \quad (3)$$

$$-8hf + 4g(c - a) = 3\lambda \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$-4f^2 + (c - a)^2 = \lambda. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The sum of these gives $4g^2 + 4g(c + a) + (c + a)^2 = 0$,

hence [illegible] $= [illegible]$. The second added to the fourth gives [illegible] $= -$ [illegible].

Hence [illegible].

The three equations that make up the five can now be written—

$$h^2 - [illegible] = 4[illegible] \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$[illegible] = -2[illegible] \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$-4[illegible] = 2[illegible] \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

that is, $h^2 = 4[illegible]$,

$4f^2 = [illegible] - 2[illegible] + c)^2$

$= -[illegible] - 4ac$,

$2[illegible] = [illegible] - 2[illegible] + c)^2$.

The two values for $4h^2f^2$, namely, $(2[illegible])^2$ and $h^2 \times 4f^2$, obtained from these give—

$$[4(a + c)^2 - [illegible]][-(a + c)^2 - 4ac] = [[illegible](a + c) - 2(a + c)^2]^2,$$

from which—

$$[2(a + c) - a][2(a + c) + a][-(a + c)^2 - 4ac] = (a + c)^2[a - 2(a + c)]^2.$$

Since $2(a + c) - a$ is a factor of both sides, one solution is $2(a + c) - a = 0$, that is, $a + 2c = 0$, $a = -2c$.

This gives $2g = 2c - c = c$, and $\lambda = 2c^2$.

Hence $h^2 = 4(a + c)^2 - a^2 = 4c^2 - 4c^2 = 0$,

$4f^2 = -(a + c)^2 - 4ac = -c^2 + 8c^2 = 7c^2$.

Since c is a factor in all, it divides out; or, we may take for c any value we please, *e. g.* -1. Then—

$$a = 2,\ h = 0,\ 2g = -1,\ 2f = \pm\sqrt{7}.$$

Choose for f the positive value, then the rectangular hyperbola is—

$$2x^2 - 2y^2 - x + \sqrt{7}\,.\,y - 1 = 0.$$

The centre is at $\left(\frac{1}{4}, \frac{\sqrt{7}}{4}\right)$; when the origin is transferred to this point the equation becomes $x^2 - y^2 = \frac{1}{8}$. Hence

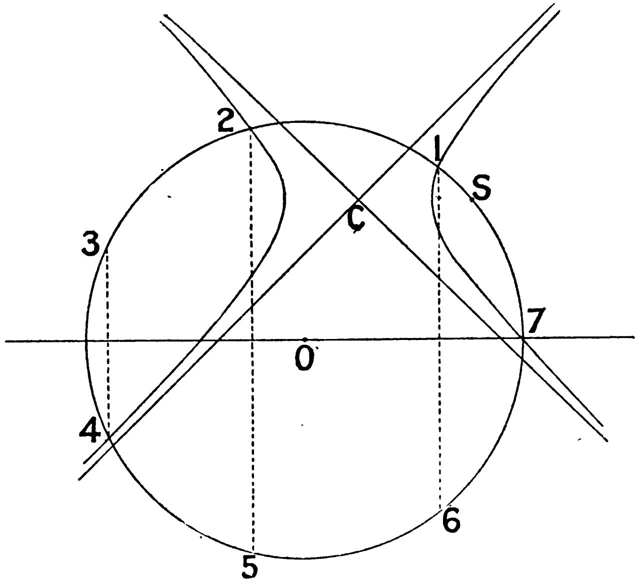

Fig. 99.

the semi-axis major is $\frac{1}{4}\sqrt{2}$. The curve lies as shown in Fig. 99; it passes through the vertices 1, 2, 4, 7. Note that one focus, S, is $\left(\frac{3}{4}, \frac{\sqrt{7}}{4}\right)$, a point on the circle; and

that the other directrix, which is $x = \frac{1}{4} - \frac{1}{2\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = 0$, passes through the centre of the circle.

The pencil of conics through these four vertices is—

$$2x^2 - 2y^2 - x + \sqrt{7}\, y - 1 + \lambda(x^2 + y^2 - 1) = 0.$$

206. *Example* 27.—Find conics that shall meet a given circle at four vertices of a regular nonagon.

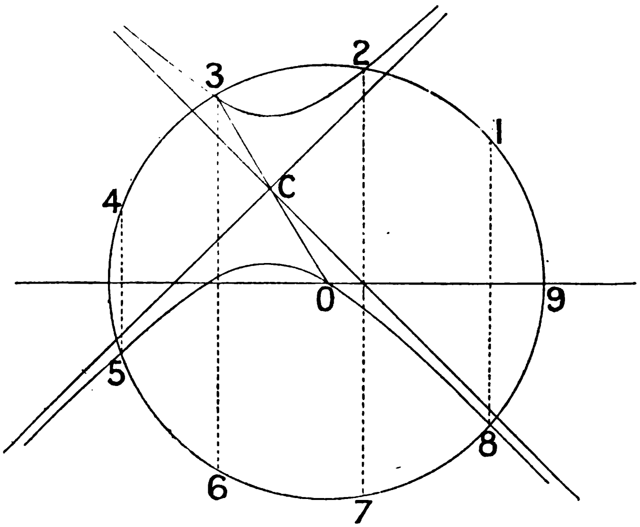

Fig. 100.

Here $$3\alpha = \frac{2\pi}{3}, \quad \cos 3\alpha = -\frac{1}{2},$$

hence $$4\cos^3 \alpha - 3\cos\alpha + \frac{1}{2} = 0,$$

that is, if $x = \cos\alpha$, $\quad 8x^3 - 6x + 1 = 0.$

The roots of this equation are the abscissæ (Fig. 100) of the vertices 1 (and 8), 4 (and 5), 7 (and 2). To bring in 3 (and 6), for which $x = -\frac{1}{2}$, multiply by $2x + 1$, and the equation becomes $16x^4 + 8x^3 - 12x^2 - 4x + 1 = 0$.

By the same argument as before, the coefficients in the equation of a rectangular hyperbola through four vertices satisfy the equations—

$$4h^2 + 4a^2 = 16\lambda \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$8hf + 8ag = 8\lambda \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$4f^2 - 4h^2 + 4g^2 + 4a(c - a) = -12\lambda \quad . \quad . \quad . \quad . \quad (3)$$

$$-8hf + 4g(c - a) = -4\lambda \quad . \quad . \quad . \quad . \quad (4)$$

$$-4f^2 + (c - a)^2 = \lambda \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The sum of these gives $4g^2 + 4g(a + c) + (a + c)^2 = 9\lambda$; the second added to the fourth gives $4g(a + c) = 4\lambda$.

Hence $$(a + c)^2 - 5g(a + c) + 4g^2 = 0,$$
$$(a + c - g)(a + c - 4g) = 0.$$

Since only one solution is desired, take $a + c - 4g = 0$ (it is found by trial that this yields simpler results than $a + c - g = 0$); then $4g = a + c$, $\lambda = \frac{1}{4}(a + c)^2$.

Hence
$$h^2 + a^2 = (a + c)^2,$$
$$4hf + a(a + c) = (a + c)^2,$$
$$-16f^2 + 4(a - c)^2 = (a + c)^2.$$

Since $(4hf)^2 = 16f^2 \times h^2$,

$$c^2(a + c)^2 = [4(a - c)^2 - (a + c)^2][(a + c)^2 - a^2]$$
$$= [4(a - c)^2 - (a + c)^2]c(2a + c).$$

Hence $c[c^3 - 3ac^2 - 9a^2c + 3a^3] = 0$.

The solution $c = 0$ gives $g = \frac{a}{4}$, $h = 0$, $f = \pm\frac{\sqrt{3}}{4}a$. Hence the rectangular hyperbola is—

$$ax^2 - ay^2 + \frac{a}{2}x + \frac{\sqrt{3}}{2}ay = 0,$$

that is, $$2x^2 - 2y^2 + x + \sqrt{3}\,.\,y = 0.$$

The centre is $\left(-\frac{1}{4}, \frac{\sqrt{3}}{4}\right)$; when this is taken as origin, the equation becomes—

$$-x^2 + y^2 = \frac{1}{8}.$$

Hence the semi-axis major is $\frac{1}{4}\sqrt{2}$. The curve lies as shown in Fig. 100; it passes through the vertices 2, 3, 5, 8.

The pencil of conics through these four vertices is—

$$2x^2 - 2y^2 + x + \sqrt{3}\,y + \lambda(x^2 + y^2 - 1) = 0.$$

By a similar process it can be shown that the circle is cut at four vertices of a regular pentagon by the rectangular hyperbola $2x^2 - 2y^2 + 2x + 1 = 0$.

207. If the origin is taken at the centre of the hyperbola, and the scale changed so that the hyperbola is $x^2 - y^2 + 1 = 0$, the circle which cuts the curve in four vertices of a regular pentagon is $(x - 1)^2 + y^2 = 4$; for the regular heptagon (the axes of x and y must be interchanged to obtain the hyperbola as here given) the circle is—

$$\left(x - \sqrt{\frac{7}{2}}\right)^2 + \left(y + \frac{1}{\sqrt{2}}\right)^2 = 8;$$

and for the regular nonagon it is—

$$\left(x - \frac{1}{\sqrt{2}}\right)^2 + \left(y + \sqrt{\frac{3}{2}}\right)^2 = 8.$$

EXAMPLES.

1. Give a graphical solution of the problem of the duplication of the cube; that is, find x to satisfy $x^3 = 2a^3$.

2. Give a graphical solution of the problem of finding two mean proportionals between a and b; that is, find x, y so that $a : x = x : y = y : b$.

3. Form the equation of the rectangular hyperbola that has for diameter a radius of $x^2 + y^2 = 1$ of slope $\tan 2\theta$, and meets the circle also at points whose ordinates are the roots of $4y^3 - 3y^2 + h = 0$. Hence show that any rectangular hyperbola of the pencil $u + \lambda v = 0$,

where
$$u = (x^2 - y^2)\cos a + 2xy \sin a - x \cos a - y \sin a,$$
$$v = -(x^2 - y^2)\sin a + 2xy \cos a - x \sin a + y \cos a,$$

serves to trisect the angle a.

Construct the hyperbolas for which (i) the asymptotes, (ii) the axes, are parallel to the axes of coordinates.

4. Prove the following construction for the trisection of an angle AOB at the centre of a circle—

The diameter AO meets the circle again at A′; take the angle A′OP = 2AOB, measured in the opposite direction to AOB. Bisect OP at C, draw through C lines parallel and perpendicular to OA. A rectangular hyperbola with these lines as asymptotes, passing through O and P, meets the circle at one point of trisection of the angle AOB.

5. Find the equations of the parabolas by which four vertices of a regular heptagon are determined. Draw these parabolas.

6. Find the equations of the parabolas by which four vertices of a regular nonagon are determined. Draw these parabolas.

7. Prove that a rectangular hyperbola is cut at four vertices of a regular pentagon by a circle on the minor axis as radius.

8. Prove that a rectangular hyperbola is cut at four vertices of a regular heptagon by a circle of radius equal to the distance between the foci, with its centre on one directrix, and passing through the other focus.

9. Prove that a rectangular hyperbola is cut at four vertices of a regular nonagon by a circle which has for radius a diameter of the hyperbola of length equal to the distance between the foci.

10. Find the circles that cut $y^2 = x$ at four vertices of a regular pentagon, heptagon, and nonagon.

11. Find the parabolas that cut $x^2 - y^2 + 1 = 0$ at four vertices of a regular pentagon, heptagon, and nonagon.

12. Find the rectangular hyperbolas that cut $y^2 = x$ at four vertices of a regular pentagon, heptagon, and nonagon.

THE END

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